5/8 Review Eigenvectors Ch7

Systems of Differential Equations

Review Exam 3: Ch1, Ch2, 3, 4, 5, 7

Ch7: Goal Given a 2×2 matrix find its eigenvectors and eigenvalues.

Method: 1) Solve the characteristic polynomial to find eigenvalues

2) For each eigenvalue find an eigenvector.

Ex) Find eigenvectors and eigenvalues of

$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \longrightarrow$ 1) $\boxed{\begin{vmatrix} (1-\lambda) & 5 \\ 2 & (4-\lambda) \end{vmatrix} = 0}$

$$(1-\lambda)(4-\lambda) - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

There are two eigenvalues $\lambda_1 = 6$, $\lambda_2 = -1$

2) $\lambda_1 = 6$ $\begin{bmatrix} -5 & 5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} -5x_1 + 5x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases}$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 6$

$\lambda_2 = -1$ $\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} 2x_1 + 5x_2 = 0 \\ 2x_1 + 5x_2 = 0 \end{cases}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$.

What can go wrong? Answ: Characteristic equation may have complex roots or double roots.

Application: Differential Equations.

Ex 1) Solve $\frac{dx}{dt} = -3x$ so that $x(0)=9$

$$\boxed{\frac{dx(t)}{dt} = -3x(t)}$$

Solution: $x(t) = Ce^{-3t}$ to find C

use $9 = x(0) = Ce^{-3(0)} = C$

Answer: $\boxed{x(t) = 9e^{-3t}}$

Check: $x' \stackrel{?}{=} -3x$

$$\begin{array}{ccc} x' & \stackrel{?}{=} & -3x \\ \| & & \| \\ -27e^{-3t} & & -27e^{-3t} \end{array}$$

Check: Initial Value $9 = x(0) = 9 \cdot 1 = 9$

Formula: ALL SOLUTIONS

to $x' = ax$ are $x(t) = Ce^{at}$

$\frac{dx}{dt} = ax$

Ex2) Solve a linear 2x2 differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ if $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$

SAME

$$\begin{bmatrix} x_1'(t) = 2x_1(t) \\ x_2'(t) = 3x_1(t) - x_2(t) \end{bmatrix} \overset{\text{SAME}}{\longleftrightarrow} \begin{cases} x_1' = 2x_1 \\ x_2' = 3x_1 - x_2 \end{cases}$$

1) verify $\vec{x} = Ce^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution for any C = const.

$$\boxed{\begin{matrix} x_1 = Ce^{2t} & x_2 = Ce^{2t} \\ x_1' = 2Ce^{2t} & x_2' = 2Ce^{2t} \end{matrix}}$$

$\lambda = 2$ is an eigenvalue with corresp eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) verify $\vec{x} = De^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution for any D.

$$\boxed{\begin{matrix} x_1 = 0 & x_2 = De^{-t} \\ x_1' = 0 & x_2' = -De^{-t} \end{matrix}}$$

$\lambda = -1$ is an eigenvalue with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ALL SOLUTIONS:

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

☆ Verify that $\vec{x}$ solves the D.E $\vec{x}' = A\vec{x}$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$\frac{d\vec{x}}{dt} = 2c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = A\left(c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= c_1 e^{2t} \underbrace{A\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{= \begin{bmatrix} 2 \\ 2 \end{bmatrix}} + c_2 e^{-t} \underbrace{A\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

$$A\vec{x} = c_1 e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Ex) Solve IVP = D.E + Initial Condition

$$\begin{cases} x_1' = -3x_1 + 2x_2 \\ x_2' = 2x_1 - 3x_2 \end{cases}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

solution:

1. $$\begin{vmatrix} (-3-\lambda) & 2 \\ 2 & (-3-\lambda) \end{vmatrix} = 0 \longleftrightarrow (-3-\lambda)(-3-\lambda)-4=0$$

$$\lambda^2+6\lambda+5=0$$

$$(\lambda+5)(\lambda+1)=0$$

2. $\boxed{\lambda_1 = -5, \quad \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{cases} 2x_1+2x_2=0 \\ 2x_1+2x_2=0 \end{cases}$$

$\boxed{\lambda_2 = -1, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -2x_1+2x_2=0 \\ 2x_1-2x_2=0 \end{cases}$$

ALL SOLUTIONS TO $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$

are $\vec{x} = c_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix}4\\2\end{bmatrix} = \vec{x}(0) = c_1 e^0 \begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^0 \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{cases} 4 = c_1 + c_2 \\ 2 = -c_1 + c_2 \end{cases}$$

$$6 = 2c_2 \rightarrow c_2 = 3,\ c_1 = 1$$

SOLUTION: $\vec{x} = e^{-5t}\begin{bmatrix}1\\-1\end{bmatrix} + 3e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

or

$$\boxed{\begin{aligned} x_1(t) &= e^{-5t} + 3e^{-t} \\ x_2(t) &= -e^{-5t} + 3e^{-t} \end{aligned}}$$

$x_1' =$

$x_2' =$

Exam Review: Dec 2006

#1) $A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

a) find ~~the~~ A^{-1}

b) use answer (a) to solve $\begin{cases} 2x + 2z = 2 \\ 2x + y + z = 5 \\ 3x + 2y + 2z = 8 \end{cases}$

c) solve for x in (b) using Cramer's

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a) $$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 0 & 1 & \underbrace{\frac{1}{2} \quad -2 \quad 1}_{A^{-1}} \end{array}\right]$$

b) $$\begin{bmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$A^{-1} A \vec{x} = A^{-1} \vec{b}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{x} = A^{-1} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

c) $$X = \frac{\begin{vmatrix} 2 & 0 & 2 \\ 5 & 1 & 1 \\ 8 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{vmatrix}} = \frac{2 \cancel{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}^{6} + 2 \begin{vmatrix} 5 & 1 \\ 8 & 2 \end{vmatrix}}{2 \cancel{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}^{0} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}}$$

$$= \frac{4}{2} = 2$$

#2) Solve $\begin{cases} y_1' = y_1 - 2y_2 \\ y_2' = -2y_1 + y_2 \end{cases}$

subject to initial conditions $y_1(0) = 1$
$y_2(0) = 3$.

Solution

(1) Solve $\begin{vmatrix} (1-\lambda) & -2 \\ -2 & (1-\lambda) \end{vmatrix} = 0$

$$(1-\lambda)^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda - 3)(\lambda + 1) = 0$$

$\boxed{\begin{matrix} \lambda_1 = 3 \\ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{matrix}}$ $\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \longrightarrow \begin{matrix} -2y_1 - 2y_2 = 0 \\ -2y_1 - 2y_2 = 0 \end{matrix}$

$\boxed{\begin{matrix} \lambda_2 = -1 \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}}$ $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\longrightarrow$ Solution $\boxed{\vec{y}(t) = c_1 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

To find c_1, c_2 use the initial conditions

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \vec{y}(0) = c_1 e^0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} 1 = -c_1 + c_2 \\ 3 = c_1 + c_2 \end{cases} \qquad \boxed{c_2 = 2 \quad c_1 = 1}$$

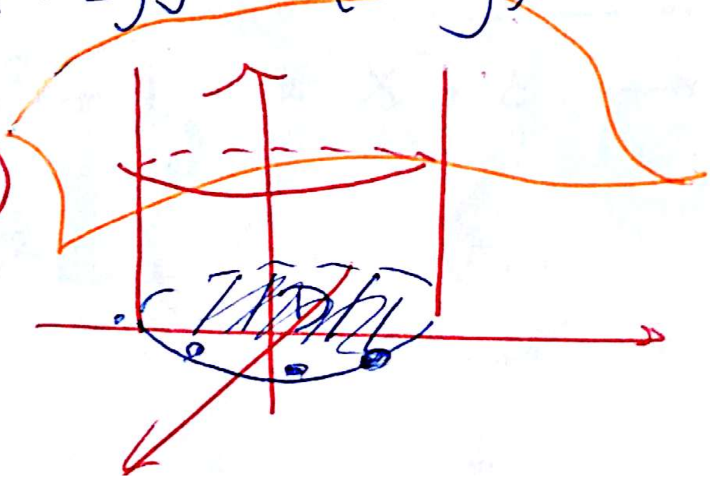
$$\boxed{\text{Final Answer } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

OR $y_1(t) = -e^{3t} + 2e^{-t}$, $y_2(t) = e^{3t} + 2e^{-t}$

#5 a) Find the area of the surface $z = xy - 1$ that lies inside the cylinder $x^2 + y^2 = 4$

b) Find $f(x,y,z)$ with $\nabla f = (2xy + 2xz)\hat{i} + (x^2 + z)\hat{j} + (x^2 + y)\hat{k}$

5a)



Solution

5a) $$\iint_D \underbrace{\sqrt{z_x^2+z_y^2+1}\,dA}_{dS}$$

$$= \iint_D \sqrt{y^2+x^2+1}\,dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \sqrt{r^2+1}\,r\,dr\,d\theta$$

$$= 2\pi \cdot \frac{(r^2+1)^{3/2}}{3}\Bigg|_{r=0}^{r=2} = \frac{2\pi}{3}\left[5^{3/2}-1\right]$$

b) $$\begin{cases} f_x = 2xy+2xz \rightarrow f = x^2y+x^2z+h(y,z) \\ ② f_y = x^2+z \qquad f = x^2y+x^2z+zy+g(z) \\ ③ f_z = x^2+y \end{cases}$$

ANSWER:
$$\boxed{f(x,y,z)=x^2y+x^2z+zy+C}$$

② $x^2+h_y = x^2+z \longrightarrow h_y = z \rightarrow h = zy + g(z)$

③ $x^2+y+g'(z) = x^2+y \qquad g(z)=C$

#6 a) Let S be surface $z = \sqrt{x^2+y^2}$ and $P(3,4,5)$

i) Find the normal vector to S at P

ii) Find an equation of tangent plane to S at P.

7) Let C be $\vec{r}(t) = \sqrt{2}\,\hat{i} + \cos^2 t\,\hat{j} + \sin^2 t\,\hat{k}$ $0 \le t \le \pi/2$ Find Length C.

Solution 6: i) $\vec{n} = \langle -z_x, -z_y, 1 \rangle$

$$= \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle = \left\langle -\frac{3}{5}, -\frac{4}{5}, 1 \right\rangle$$

ii) $\hat{n} \cdot \langle x-3, y-4, z-5 \rangle = 0$

$$\boxed{-\frac{3}{5}(x-3) - \frac{4}{5}(y-4) + (z-5) = 0}$$

Solution 7: $L(C) = \int_{t=0}^{t=\pi/2} |\vec{r}\,'(t)|\,dt$

$\vec{r}\,'(t) = \langle 0, -2\cos t \sin t, 2\sin t \cos t \rangle$

$$|\vec{r}'(t)| = \sqrt{8\sin^2 t \cos^2 t} = 2\sqrt{2}\sin t \cos t$$

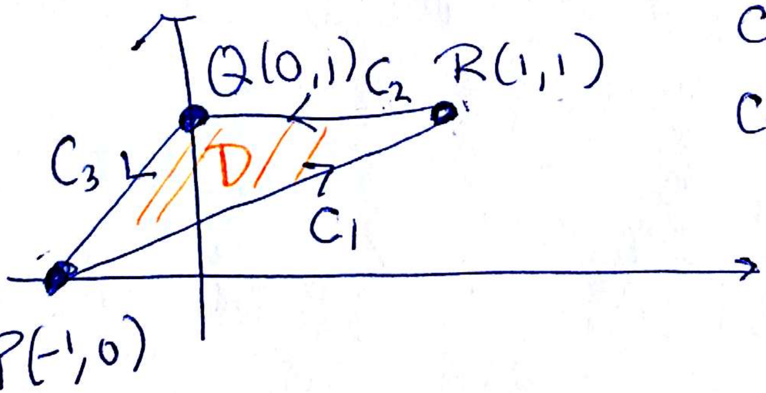
$$L(C) = \int_{t=0}^{t=\pi/2} 2\sqrt{2}\sin t \cos t \, dt$$

$$= 2\sqrt{2}\,\frac{\sin^2 t}{2}\Big|_{t=0}^{t=\pi/2} = \sqrt{2}$$

8. Let C bound the triangle with vertices $P(-1,0)$, $Q(0,1)$, $R(1,1)$, oriented counterclockwise. Evaluate $\int_C y^2 dx - x^2 dy$

a) directly as line integral

b) as a double integral using Green's.



C_1: $y = \frac{1}{2}x + \frac{1}{2}$

C_3: $y = x + 1$

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C_1: $\vec{r}_1 = \vec{OP} + t\vec{PQ} = \langle -1,0\rangle + t\langle 2,1\rangle$

$x = -1+2t \qquad dx = 2\,dt$

$y = t \qquad dy = dt$

$$\int_C y^2dx - x^2dy = \int_{t=0}^{t=1} \left[2t^2 - (2t-1)^2\right] dt$$

$$= \int_{t=0}^{t=1} (-2t^2+4t-1)\,dt = \frac{-2t^3}{3} + 2t^2 - t \Big|_{t=0}^{t=1} = \frac{1}{3}$$

C_2: $x = x \qquad y = 1$

$dx = dx \qquad dy = 0$

$$-\int_{x=0}^{x=1} 1^2\,dx - 0 = -1$$

C_3: $\vec{r}_3(t) = \vec{OQ} + t\vec{QR} = \langle 0,1\rangle + t\langle -1,-1\rangle$

$x = -t \qquad y = 1-t \qquad 0 \le t \le 1$

$dx = -dt \qquad dy = -dt$

$$\int_{t=0}^{t=1} \left[-(1-t)^2 + t^2\right] dt = \int_{t=0}^{t=1} (2t-1)\,dt = 0$$

$$\oint_C y^2dx - x^2dy = -2/3$$

b) $\int_C y^2 dx - x^2 dy = \iint_D (Q_x - P_y)\, dA$

$\boxed{P = y^2 \quad Q = -x^2}$ $= \iint_D (-2x - 2y)\, dA$

$$= \int_{y=0}^{y=1} \int_{x=(C_3)\ y-1}^{x=(C_1)\ 2y-1} (-2x-2y)\, dA = -2/3$$

Integrate (2x)

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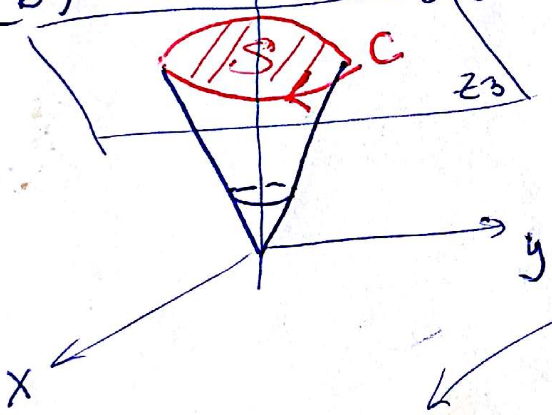
Fall 05 #9) Let C be the curve of intersection of cone $x^2+y^2=z^2$ and plane $z=3$

$\vec{F} = y\hat{i} + z\hat{j} - x\hat{k}$. Let C be oriented clockwise from above. Calculate $\oint_C \vec{F}\cdot d\vec{r}$

(a) directly as a line integral

(b) as a double integral using Stokes's.



when $z=3$ $\quad x^2+y^2=3^2$

$x = 3\cos t \qquad dx = -3\sin t\, dt$

$y = 3\sin t \qquad dy = 3\cos t\, dt$

$z = 3 \qquad dz = 0$

a) $\oint_C y\,dx + z\,dy - x\,dz$

$$= \int_{t=0}^{t=2\pi} \Big(-3^2\underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}} + \underbrace{\cancel{3^2\cos t}}_{0}\Big)\,dt$$

$$= 3^2\int_{t=0}^{t=2\pi} \frac{1}{2} - \underbrace{\cancel{\frac{\cos 2t}{2}}}_{0}\,dt = 9\pi$$

b) $\oint_C y\,dx + z\,dy - x\,dz = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$, $\quad d\vec{S} = \hat{n}\,dS$

$\hat{n} = -k \qquad \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & -x \end{vmatrix} = \langle -1, 1, -1 \rangle$

$$= \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \iint_S 1\cdot dS = \text{Area}(S)$$

$$= \pi\cdot 3^2$$

Fall 05 #10) Let T be Volume $\begin{cases} x^2+y^2+z^2 \le 4 & (S_1) \\ z \ge 1 & (S_2) \end{cases}$

S = two part boundary of T

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$. Use outward normal to compute $\oiint_S \vec{F}\cdot d\vec{S}$

a) directly as a surface integral

b) as a triple integral using divergence theorem.

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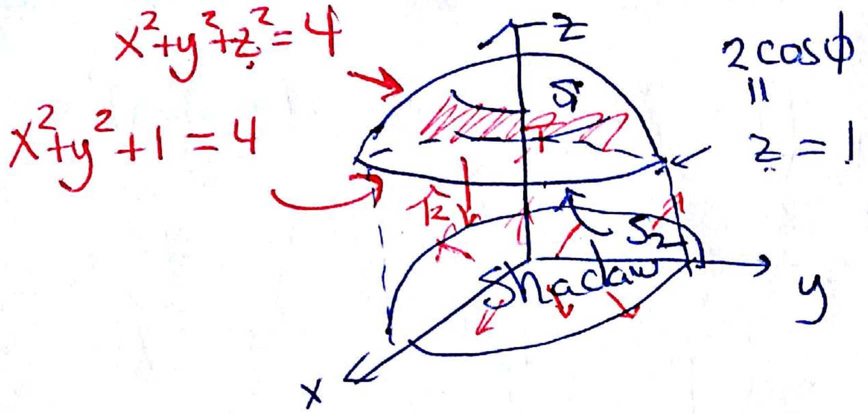

$\cos\phi = 1/2$

$\phi = \pi/3$

a) $$\iint_{S_1} \vec{F}\cdot \underbrace{d\vec{S}}_{=\hat{n}\,dS} = \iint_{S_1} \underbrace{\langle x,y,z\rangle}_{\vec{F}} \cdot \frac{\langle x,y,z\rangle}{2}\, dS = \iint_{S_1} \frac{x^2+y^2+z^2}{2}\, dS$$

$$= 2\iint_{S_1} 1\, dS = 2\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/3} 2^2 \sin\phi \, d\phi\, d\theta$$

$$= 2^3\cdot 2\pi \left(-\cos\phi\right)\Big|_0^{\pi/3} = 8\pi$$

b) $$\iint_{S_2} \vec{F}\cdot \overset{-\hat{k}}{\hat{n}}\, dS = \iint_{S_2} \underbrace{\langle x,y,z\rangle\cdot\langle 0,0,-1\rangle}_{-z=-1} dS$$

$$= -\iint_{S_2} dS = -\text{Area}(S_2) = -\pi(\sqrt{3})^2 = -3\pi$$

$$\oiint_{S} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = 8\pi + (-3\pi) = 5\pi$$

b) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_T \nabla \cdot \vec{F}\, dV = 3 \iiint_T dV$

$\boxed{\nabla \cdot \langle x, y, z \rangle = 1+1+1 = 3}$

$= 3 \cdot \text{Volume}(T)$

$$= 3 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{3}} \int_{z=1}^{z=\sqrt{4-r^2}} 1 \, dz\, r\, dr\, d\theta = 6\pi \int_{r=0}^{r=\sqrt{3}} \left(\sqrt{4-r^2} - 1\right) \cdot r\, dr$$

$$= 6\pi \left[\frac{(4-r^2)^{3/2}}{-3} - \frac{r^2}{2} \right] \Bigg|_{r=0}^{r=\sqrt{3}} = 6\pi \left(\left[-\frac{1}{3} - \frac{3}{2} \right] - \left[\frac{-8}{3} - 0 \right] \right)$$

$$= \cancel{6}\pi \left(\frac{-2 - 9 + 16}{\cancel{6}} \right) = 5\pi$$

Spring 05 #1a)

Write system whose augmented matrix is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 1 & 3 & -2 & 0 & 0 & -1 & 0 \\ 2 & 6 & -4 & 1 & 0 & 0 & 0 \\ 1 & 3 & -2 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{cases} x_4 + x_6 = -2 \\ x_1 + 3x_2 - 2x_3 - x_6 = 0 \\ 2x_1 + 6x_2 - 4x_3 + x_4 = 0 \\ x_1 + 3x_2 - 2x_3 + x_4 + x_6 = 0 \end{cases}$$

Then use Gaussian Elimination to solve and write your answer as a linear combination of vectors.

$$\rightarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right] \end{array}$$

$$\rightarrow \begin{array}{c} \begin{array}{cccccc} x_1 & & & x_4 & & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{array}\right] \end{array}$$

$$\rightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

$x_1 = 2 - 3x_2 + 2x_3$

$x_4 = -4$

$\rightarrow x_6 = 2$

pivots in terms of free's

all solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2-3x_2+2x_3 \\ x_2 \\ x_3 \\ -4 \\ x_5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -4 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ -4 \\ 0 \\ 2 \end{bmatrix}}_{\text{particular solution}} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{linear combination of the null solution } \left(A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)}$$

for ANY x_2, x_3, x_5

TOTAL SOLUTION