5/1 Ch4 Determinants, Adjoint Matrix

Ch5 Cramer's Rule

Ex) #5 Spring 2008 Final

calculate det A when $A = \begin{bmatrix} 0 & -2 & -2 & 4 \\ \boxed{1} & 4 & 6 & -3 \\ 1 & 6 & -4 & 5 \\ 2 & 12 & 1 & 6 \end{bmatrix}$

$$\det(A) = |A| = 2\begin{vmatrix} 1 & 6 & -3 \\ 1 & -4 & 5 \\ 2 & 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 & -3 \\ 1 & 6 & 5 \\ 2 & 12 & 6 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 & 6 \\ 1 & 6 & -4 \\ 2 & 12 & 1 \end{vmatrix}$$

Too much computation

Start over
$$|A| = \begin{vmatrix} 0 & -2 & -2 & 4 \\ 1 & 4 & 6 & -3 \\ 0 & 2 & -10 & 8 \\ 0 & 4 & -11 & 12 \end{vmatrix} \begin{array}{l} R1 \\ R2 \\ R3-R2 \\ R4-2R2 \end{array}$$

$$= 0 \cdot \left| \quad \right| - 1 \cdot \begin{vmatrix} -2 & -2 & 4 \\ 2 & -10 & 8 \\ 4 & -11 & 12 \end{vmatrix} + 0 \begin{vmatrix} -2 & -2 & 4 \\ 4 & 6 & -3 \\ 4 & -11 & 12 \end{vmatrix} - 0 \left| \quad \right|$$

$$= -\begin{vmatrix} -2 & -2 & 4 \\ 0 & -12 & 12 \\ 0 & -15 & 20 \end{vmatrix} = +2\begin{vmatrix} -12 & 12 \\ -15 & 20 \end{vmatrix} = +2(-12 \cdot 20 + 12 \cdot 15)$$

$$= 24(-20+15)$$
$$= 24(-5) = -120$$

Ex) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ Find $|A|$

I. Use Laplace Expansion $|A| = 2\cdot 5 - 3\cdot 4 = -2$

II. Use Laplace + Gaussian Elimination

$$|A| \underset{\text{G.E}}{=} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \underset{\text{Laplace}}{=} 2(-1) - 3(0) = -2$$

Ex2) $A = \begin{bmatrix} \overset{+}{2} & \overset{-}{4} & \overset{+}{6} \\ -2 & +1 & -0 \\ +1 & -0 & +1 \end{bmatrix}$

Method I Laplace Expansion:

2nd Row
$$|A| = -2 \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} ? \end{vmatrix}$$

$$= -2(4) + 1(2-6) = -12$$

3rd column
$$|A| = 6 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} ? \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix}$$

$$= 6(-1) + 1(2-8) = -12$$

$$|A| = 1 \cdot \begin{vmatrix} 1 & -2 & -2 \\ 1 & -2 & 1 \\ -4 & 4 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & -2 \\ -4 & 4 \end{vmatrix}$$

$$= 1 \cdot (-2-4) + 2(1+4) - 2(4-8) = -6 + 10 + 8$$

$$= 12$$

ii) $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{12}$

iii) $|2A^3 A^T| = 2^4 \cdot \det(A) \cdot \det(A) \cdot \det(A) \cdot \det(A^T)$

$$= 2^4 \cdot 12^4$$

b) i) $BC = 1(-2) + 2(3) + -1(1) = -2 + 6 - 1$

$$= 3$$

ii) $B^T C^T = \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} -2 & 3 & 1 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} -2 & 3 & 1 \\ -4 & 6 & 2 \\ 2 & -3 & -1 \end{bmatrix}}$

2^{nd} Method to find Inverse

If $\det A \neq 0$ then $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

Ex) Find the inverse $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ using two methods.

$\underset{A_{11}}{2}$

I. Gaussian Elimination: $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc} \boxed{1} & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \boxed{2} & 0 & 1 \\ 0 & +1 & -1 & +2 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$ with $\underbrace{\begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix}}_{A^{-1}}$ $\qquad A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

II. Use Adjoints $|A| = 4 - 3 = 1$

$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \operatorname{adj} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Ex) use adjoint method to find A^{-1}

when $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

$$|A| = 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} ? \end{vmatrix} + 2 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -3 + 2$$

$$= -1$$

$$\text{adj} A = \begin{vmatrix} -2 & -4 & -3 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}^T = \begin{bmatrix} -2 & 0 & 1 \\ -4 & 1 & 2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$

Method II. Use Gaussian Elimination + Laplace Expansion to compute $\begin{vmatrix} 2 & 4 & \boxed{6} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -4 & 4 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 \\ R_3 \end{matrix} = 1 \cdot \begin{vmatrix} -4 & 4 \\ 2 & 1 \end{vmatrix} = -4-8$$
$$= -12$$

Ex) Spring 2006 #1a)

$$A = \begin{bmatrix} 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & -1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \quad |A| = \begin{matrix} \\ \\ \\ R_4 - 3R_3 \end{matrix} \begin{vmatrix} 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & -4 & 4 & 1 \end{vmatrix}$$

a) Find determinants of i) A ii) A^{-1} iii) $2A^3A^T$

b) $B = [1 \quad 2 \quad -1]$ $C = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ Find

i) BC ii) $B^T C^T$

Ch4, Ch5 Cramer's Rule

5/3 Ch7 Eigenvalues

Review $A^{-1} = \frac{1}{\det A} \cdot \text{adj}(A)$

$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$ This leads to

Cramer's Rule. Solve $\boxed{A\vec{x} = \vec{b}}$ $= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Ex) $A = \begin{bmatrix} 1 & -1 & \boxed{1} \\ 2 & -1 & \boxed{0} \\ 1 & -2 & \boxed{2} \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 2 & -1 & 0 & b_2 \\ 1 & -2 & 2 & b_3 \end{array}\right]$

$|A| = 1 \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 2 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -3 + 2 = -1$

since $|A| \neq 0$, there are 3 pivots $\overset{\text{SAME}}{\Longleftrightarrow}$ rank$(A) = 3$ $\overset{\text{SAME}}{\Longleftrightarrow}$ no free variables. $\overset{\text{SAME}}{\Longleftrightarrow}$ The square $A\vec{x} = \vec{b}$ has 1 solution.

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \text{adj}(A) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}$$

Ex Use Cramer's to solve $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$\begin{cases} 2x_1 + 3x_2 = 4 \\ 4x_1 + 5x_2 = 6 \end{cases}$$

Cramers $x_1 = \dfrac{\begin{vmatrix} 4 & 3 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}} = \dfrac{2}{-2} = -1$

$$x_2 = \frac{\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}} = \frac{-4}{-2} = 2$$

Ex (Spring 2005) Use Cramer's Rule to solve for w ONLY when

$$\begin{cases} 2x + 4y \quad + 2w = 0 \\ x + 3y \quad + w = 0 \\ x + 4y + z + w = 1 \\ 3x \quad + 4z + w = 2 \end{cases}$$

by Cramer's

$$w = \frac{\begin{vmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 3 & 0 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 0 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 4 & 1 & 1 \\ 3 & 0 & 4 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 1 & -8 & 2 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 0 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 4 & 1 & 1 \\ -1 & -16 & 0 & -3 \end{vmatrix}}$$

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$$w = \frac{-1 \cdot \begin{vmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 1 & -8 & 2 \end{vmatrix}}{1 \cdot \begin{vmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ -1 & -16 & -3 \end{vmatrix}} = \frac{-2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}}{2 \cdot \begin{vmatrix} 3 & 1 \\ -16 & -3 \end{vmatrix} - 4 \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ -1 & -16 \end{vmatrix}}$$

$$= \frac{-4}{14+8-26} = \frac{-4}{-4} = 1$$

Ex (Fall 2005) a) Solve $\begin{cases} x + y + z = 0 \\ -4x + 2y - z = 3 \\ -5x + y - 2z = 3 \end{cases}$

b) Can we use Cramer's rule to solve the system.

$|A| = 0$ so cannot use Cramer's

Solution

a) G.E. $\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ -4 & 2 & -1 & 3 \\ -5 & 1 & -2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \boxed{6} & 3 & 3 \\ 0 & 6 & 3 & 3 \end{array}\right]$

$$\longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z = \text{free} \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{2} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Solve for pivots x, y in terms of frees z.

$$\begin{cases} x + y + z = 0 \\ 2y + z = 1 \\ 0 = 0 \end{cases}$$

$$y = \frac{1}{2} - \frac{z}{2} \qquad x = -y - z = -\frac{1}{2} - \frac{z}{2}$$

Write answer as a linear combination

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{z}{2} \\ \frac{1}{2} - \frac{z}{2} \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + z \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$z = 0 \quad \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$ ~~z ≠ ½, 0, 7, 3~~

Ch7 Eigenvalues

A vector $\vec{x} \neq \vec{0}$ is an eigenvector of a square matrix A if $A\vec{x} = \lambda \vec{x}$
λ scalar

λ = eigenvalue associated to $\vec{x}$.

Ex) $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$$

$A\vec{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\vec{x}$

$\vec{x}$ has associated eigenvalue 4.

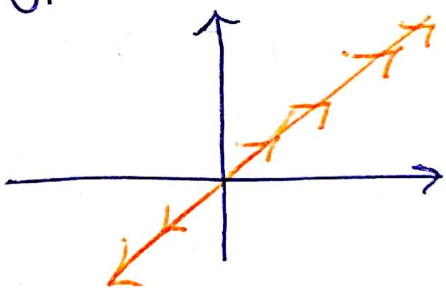
Ex2) $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is NOT an eigenvector of some A.

$A\vec{y} = \begin{bmatrix} 2 \\ -10 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda \vec{y}$

Ex3) $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\quad A\vec{v} = \begin{bmatrix} 12 \\ 12 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$= 4\vec{v}$

so $\vec{v}$ is an eigenvector with eigenvalue 4.



Ex4) $\vec{w} = \begin{bmatrix} -201 \\ -201 \end{bmatrix}$

$A\vec{w} = \begin{bmatrix} -804 \\ -804 \end{bmatrix} = 4\vec{w}$

Goal: Given a square matrix, find its eigenvalues and vectors.

$A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$

$$A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{cases}3x_1 + x_2 = \lambda x_1\\ -3x_2 + 7x_2 = \lambda x_2\end{cases}$$

$$\begin{cases}(3-\lambda)x_1 + x_2 = 0\\ -3x_2 + (7-\lambda)x_2 = 0\end{cases} \rightarrow \left[\begin{array}{cc|c}(3-\lambda) & 1 & 0\\ -3 & (7-\lambda) & 0\end{array}\right]$$

If $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ is an eigenvalue then

$$\begin{vmatrix}(3-\lambda) & 1\\ -3 & (7-\lambda)\end{vmatrix} = 0$$ "characteristic equation"

$$(3-\lambda)(7-\lambda)+3=0$$

$$\lambda^2 - 10\lambda + 21 + 3 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda-6)(\lambda-4)$$ Two eigen values

$\lambda_1 = 6$ $\lambda_2 = 4$

$\lambda_1 = 6 \longrightarrow \begin{cases} -3x_1 + x_2 = 0 \\ -3x_1 + x_2 = 0 \end{cases} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = 4 \longrightarrow \begin{cases} -x_1 + x_2 = 0 \\ -3x_1 + 3x_2 = 0 \end{cases} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Exam) Find the eigen values of $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

$\longrightarrow \begin{vmatrix} -1-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(-3-\lambda) - 4 = 0$

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