4/24 Ch 5: Inverses

Review Ch 3: Systems of Equations

Case 1: . . . $\rightarrow$ $$\left[\begin{array}{cccc|c} & & ? & & \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & -7 \\ & & ? & & \end{array}\right]$$
inconsistent

$\underbrace{}_{A = \text{coeff}}$

case 2: Not case 1 and no free variables. Then ONLY one solution.

case 3: Not case 1 and there are frees Then there will be an infinite # of solutions.

rank (A) = # of pivots

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4c) Suppose the coefficient matrix of a system is a 4x5 matrix with rank 4. Is it possible the system is inconsistent? Explain

$A = \begin{bmatrix} \quad & & & & \quad \\ & & & & \\ & & & & \end{bmatrix}$

Review Ch 1:

Ex) $\underset{2\times 2}{A=\begin{bmatrix}1&2\\3&4\end{bmatrix}}$ $\quad$ $\underset{2\times 3}{C=\begin{bmatrix}1&2&1\\0&1&2\end{bmatrix}}$

$\underset{3\times 2}{D=\begin{bmatrix}0&1\\1&0\\2&3\end{bmatrix}}$ Compute

$\underset{2\times 2\,\cdot\,2\times 3}{\underline{AC}}=\begin{bmatrix}1&4&5\\3&10&11\end{bmatrix}$ $\quad$ $CD=\begin{bmatrix}4&4\\5&6\end{bmatrix}$

$AD=$ Cannot multiply $\quad$ $\underset{2\times 3\,\cdot\,2\times 2}{\underline{CA}}=$ cannot multiply

$DC=\begin{bmatrix}0&1&2\\1&2&1\\2&7&8\end{bmatrix}$ $\quad$ $DA=\begin{bmatrix}3&4\\1&2\\11&16\end{bmatrix}$

$$A^3=A^2A=\underbrace{\begin{bmatrix}7&10\\15&22\end{bmatrix}}_{A^2}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}37&54\\81&118\end{bmatrix}$$

$\underset{2\times 2\,\cdot\,3\times 2}{AD}$ $\quad$ $AB\overset{?}{=}BA$

Ch5 Inverse Matrices

An $n\times n$ matrix A is invertible if there is an $n\times n$ matrix B so that

$$AB = I_n = BA \qquad I_n = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Ex) $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad BA = I_2$$

so B is the inverse of A, we write

$$\boxed{B = A^{-1}} \qquad \boxed{A = B^{-1}\,?}$$

Ex) Solve the system $\begin{cases} 2x + 5y = 7 \\ x + 3y = -1 \end{cases}$ by using inverses.

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \langle x, y \rangle$$

$$\vec{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\boxed{A\vec{x} = \vec{b}}$$

$$\underbrace{\begin{bmatrix}2&5\\1&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}a&c\\b&d\end{bmatrix}}_{B}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\longrightarrow\left[\begin{array}{cc|c}2&5&1\\1&3&0\end{array}\right]$$

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\longrightarrow\left[\begin{array}{cc|c}2&5&0\\1&3&1\end{array}\right]$$

$$\left[\begin{array}{c|cc}\underbrace{\begin{matrix}2&5\\1&3\end{matrix}}_{A}&\begin{matrix}1&0\\0&1\end{matrix}\end{array}\right]$$

$$\rightarrow\left[\begin{array}{cc|cc}\boxed{1}&3&0&1\\2&5&1&0\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}\boxed{1}&3&0&1\\0&+1&-1&+2\end{array}\right]$$

$$\rightarrow\left[\begin{array}{c|c}\begin{matrix}1&0\\0&1\end{matrix}&\underbrace{\begin{matrix}+3&-5\\-1&2\end{matrix}}_{A^{-1}}\end{array}\right]$$

When will this method fail?

When there are not n pivots

$$\vec{x} = \cancel{\frac{\vec{b}}{A}} \quad A^{-1}\vec{b}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 26 \\ -9 \end{bmatrix}$$

Now Solve
$$\begin{cases} 2x + 5y = -2 \\ x + 3y = 1 \end{cases}$$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\boxed{x = -11 \quad y = 4} \qquad = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

Goal: Finding Inverse Matrices

Ex) Give $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ find $B = A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Method: $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\boxed{AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = BA$$

Ex) Find A^{-1} when $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

Use A^{-1} to solve $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_2 = 2 \\ x_1 - 2x_2 + 2x_3 = 0 \end{cases}$

$$A^{-1} A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

<u>solution</u>

$$\left[\underbrace{\begin{matrix} \boxed{1} & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \right] \longrightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -2 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & +1 & +3 & -1 & -1 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array} \right]$$

$$\longrightarrow \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{matrix}}_{A^{-1}} \right] \qquad AA^{-1}$$

4/26 Ch4 Determinants

Review Problem Inverses:

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

a) find A^{-1}

b) use A^{-1} to solve $\boxed{A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b}}$

for x_1, x_2, x_3, i.e solve $\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + 2x_3 = 2 \end{cases}$

Solution

a) $$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \longrightarrow \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 & -1 \\ 0 & 2 & 3 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & -6 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 9 & -1 & -3 \\ 0 & 0 & 1 & \underbrace{-6 \quad 1 \quad 2}_{A^{-1}} \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_2 - R_3 \\ R_3 \end{array}$$

Verify $A A^{-1} = I$

$$\begin{bmatrix}1&1&1\\0&2&3\\3&2&2\end{bmatrix}\begin{bmatrix}-2&0&1\\9&-1&-3\\-6&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{b} \xleftrightarrow{\text{SAME}} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}\vec{b}$$

$$\text{So } \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}\vec{b}=\begin{bmatrix}-2&0&1\\9&-1&-3\\-6&1&2\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\2\\-1\end{bmatrix}$$

$x_1=0$ $x_2=2$ $x_3=-1$

Ch4 Determinants

Ex1) $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $\det(A)=ad-bc$

Ex2) $A=\begin{bmatrix}1&2&3\\4&5&-1\\6&8&-2\end{bmatrix} \overset{\det(A)}{=} 1\cdot\begin{vmatrix}5&-1\\8&-2\end{vmatrix}-2\begin{vmatrix}4&-1\\6&-2\end{vmatrix}+3\begin{vmatrix}4&5\\6&8\end{vmatrix}$

$=1\cdot(-10+8)-2(-8+6)+3(32-30)$

$=-2+4+6=8$

when $\det\begin{bmatrix}5&-1\\8&-2\end{bmatrix} = \begin{vmatrix}5&-1\\8&-2\end{vmatrix}$

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Properties of $\det(A)$ when A is an $n \times n$ matrix.

I. A' is obtained from A by exchanging two rows then $|A| = -|A'|$

II. A' is obtained from A by multiplying some row by the scalar c. Then
$|A'| = c\,|A|$

III A' is obtained from A by replacing a row by its sum with a multiple of another row $|A'| = |A|$

IV $|I_n| = 1$

Ex) compute $\det(A)$ when $A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Method 1 (Formula La~~grange~~place Exponsion)

$$|A| = 2\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 4\det\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + 6\det\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|A| = 2\cdot 1 - 4\cdot 2 + 6\cdot(-1) = -\cancel{8}12$$

Property $\underline{VI}$: A is square $n \times n$

$$A\vec{x} = \vec{b} \text{ has one solution} \xleftrightarrow{\text{SAME}} A \text{ has rank } n \xleftrightarrow{\text{SAME}} \det(A) \neq 0$$

Property $\underline{VII}$ $\det(AB) = \det(A)\det(B)$

Ex) Verify property $\underline{VII}$ $A = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\det A = -5$ $\det B = -2$

$AB = \begin{bmatrix} 7 & 10 \\ -15 & -20 \end{bmatrix}$ $\det(AB) = -140 + 150$

$= 10$

$= \det A \cdot \det B$

Property $\underline{VIII}$ $\det(A^T) = \det(A)$

Ex) Verify Property $\underline{VIII}$ when $A = \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -2 \\ 7 & 5 \end{bmatrix}$ $\det A = 19 = \det A^T$

Property $\underline{IX}$ $\det(A^{-1}) = \dfrac{1}{\det A}$

Method2 (Use Properties)

$$\begin{vmatrix}2&4&6\\2&1&0\\1&0&1\end{vmatrix} = -\begin{vmatrix}\boxed{1}&0&1\\2&1&0\\2&4&6\end{vmatrix} = -\begin{vmatrix}\boxed{1}&0&1\\0&\boxed{1}&-2\\0&4&4\end{vmatrix}$$

$$= -\begin{vmatrix}\boxed{1}&0&1\\0&\boxed{1}&-2\\0&0&\boxed{12}\end{vmatrix} = -\begin{vmatrix}1&0&0\\0&1&0\\0&0&12\end{vmatrix}$$

$$= -12\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} = -12$$

Property Ⅹ $\det(A) = \pm p_1 p_2 p_3 \cdots p_n$

Method3 (A different Laplace Expansion)

$$A = \left[\begin{matrix}2\\2\\1\end{matrix}\ \boxed{\begin{matrix}4\\1\\0\end{matrix}}\ \begin{matrix}6\\0\\1\end{matrix}\right] = -4\begin{vmatrix}2&0\\1&1\end{vmatrix} + 1\begin{vmatrix}2&6\\1&1\end{vmatrix} - 0\begin{vmatrix}2&6\\2&0\end{vmatrix}$$

$$= -4\cdot 2 + 1\cdot(2-6) = -12$$