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Ch 1: Matrices and Matrix Algebra

Ex) $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Ex) $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} = \left[1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} ,\ -1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$

$= \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$

Ex) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} = \left[1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 5 \end{bmatrix} ,\ \begin{matrix} 2 \\ 0 \end{matrix} ,\ \begin{matrix} 12 \\ 8 \end{matrix} \right]$

$2 \times \underline{3}$ $\qquad$ $\underline{3} \times 3$

$= \begin{bmatrix} -2 & 2 & 12 \\ -6 & 0 & 8 \end{bmatrix}$

2×3

Ex) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \end{bmatrix}$ IMPOSSIBLE

$3 \times \underline{3}$ $\qquad$ $\underline{2} \times 3$

$3 \neq 2$

Ch2 Solving Systems + Elementary Row Operations

Elementary Row Operations

1. Interchange any two rows
2. Replace any row by (nonzero) multiple of itself
3. Replace any row by itself + a multiple of a different row.

Goal of Gaussian Elimination: Do row operations until matrix is in Echelon form.

Ex) Solve $\begin{cases} x_1 + 2x_2 = 1 & (R1) \\ 2x_1 - 3x_2 = -1 & (R2) \end{cases}$ by Gaussian Elimination

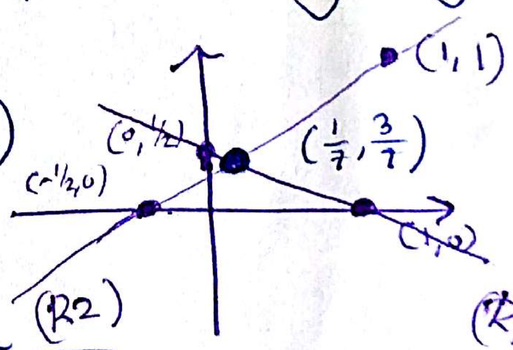
pivot = leading entry

Augment Matrix $\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 2 & -3 & -1 \end{array}\right]\begin{array}{l}(R1)\\(R2)\end{array}$ Eliminate everything under pivot = leading entry

x_1 = pivot x_2 = pivot

$\longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & \boxed{-7} & -3 \end{array}\right]\begin{array}{l}(R1)\\(R2)+(-2)(R1)\end{array}$

pivot = leading entry



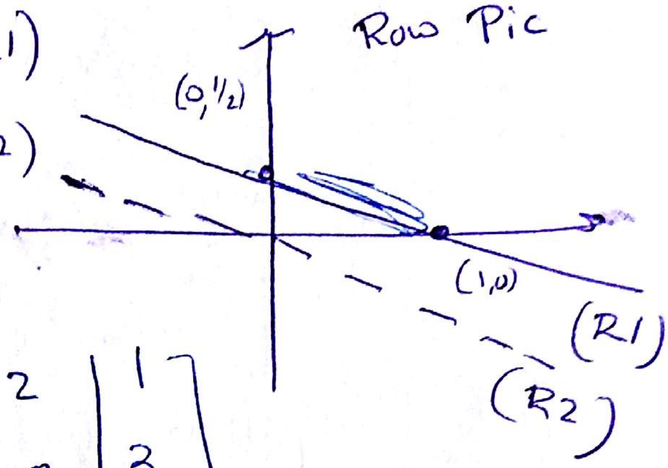
Rewrite and Backwards substitute

$\begin{cases} x_1 + 2x_2 = 1 \\ \quad -7x_2 = -3 \end{cases}$

$\boxed{\begin{array}{l} x_1 = 1/7 \\ x_2 = \frac{3}{7} \end{array}}$

Ex2) Solve $\begin{cases} x_1 + 2x_2 = 1 & (R1) \\ -3x_1 - 6x_2 = 0 & (R2) \end{cases}$

Row Pic



$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ -3 & -6 & 0 \end{array}\right] \begin{array}{c} (R1) \\ (R2) \end{array} \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

Rewrite $\begin{cases} x_1 + 2x_2 = 1 \\ 0 = 3 \end{cases}$ Inconsistent.

Ex3) Solve $\begin{cases} x_1 + 2x_2 = 1 \\ 5x_1 + 10x_2 = 5 \end{cases}$

$$\begin{array}{cc} x_1 & x_2 \end{array} \qquad \begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 5 & 10 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Rewrite $\begin{cases} x_1 + 2x_2 = 1 \\ 0 = 0 \end{cases}$

ANSWER: Solution is line $\underline{x_1 + 2x_2 = 1}$

x_1 = pivot, x_2 = free. Solve for pivot variables in terms of free variables. $x_1 = 1 - 2x_2$

solutions $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} 1-2x_2 \\ x_2 \end{bmatrix}} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

(for any x_2)

Experiment: $x_2 = 0 \rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_2 = -3 \rightarrow \vec{x} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$x_2 = 10 \rightarrow \vec{x} = \begin{bmatrix} -19 \\ 10 \end{bmatrix}$

$x_2 = \pi \quad \vec{x} = \begin{bmatrix} 1-2\pi \\ \pi \end{bmatrix}$

4/20 Ch2: Systems; Row Ops
Ch3: Varieties of Systems
Ch5: Inverse Matrices

Ex1) Solve
$$\begin{cases} 3x_1 - 6x_2 - x_3 + x_4 = 6 & (R1) \\ -x_1 + 2x_2 + 2x_3 + 3x_4 = 3 & (R2) \\ 4x_1 - 8x_2 - 3x_3 - 2x_4 = 3 & (R3) \end{cases}$$

$$\left[\begin{array}{cccc|c} \boxed{3} & -6 & -1 & 1 & 6 \\ -1 & 2 & 2 & 3 & 3 \\ 4 & -8 & -3 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \boxed{1} & -2 & -2 & -3 & -3 \\ 3 & -6 & -1 & 1 & 6 \\ 4 & -8 & -3 & -2 & 3 \end{array}\right]$$

x_2 = free

$$\rightarrow \left[\begin{array}{cccc|c} \boxed{1} & -2 & -2 & -3 & -3 \\ 0 & 0 & \boxed{5} & 10 & 15 \\ 0 & 0 & 5 & 10 & 15 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{cccc} & x_2 & \text{frees} & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & -2 & -2 & -3 & -3 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Rewrite
$$\begin{cases} x_1 - 2x_2 - 2x_3 - 3x_4 = -3 \\ x_3 + 2x_4 = 3 \\ 0 = 0 \end{cases}$$

and solve for pivot variables (x_1, x_3) in terms of free variables (x_2, x_4)

$x_3 = 3 - 2x_4 \qquad x_1 = -3 + 2x_2 + 2\underbrace{(3-2x_4)}_{x_3} + 3x_4$

All SOLUTIONS : $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 + 2x_2 - x_4 \\ x_2 \\ 3 - 2x_4 \\ x_4 \end{bmatrix}$ for any x_2, x_3

as a linear combination

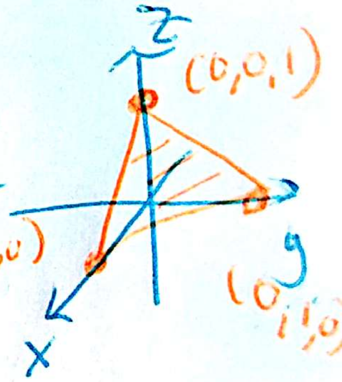
$$\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The solutions form a 2 plane in 4D.

$x_2 = 0,\ x_4 = 0 \quad \vec{x} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \qquad x_2 = 1,\ x_4 = -2 \quad \vec{x} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ -2 \end{bmatrix}$

$x_2 = -1,\ x_4 = 0 \quad \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} \qquad x_2 = \sin(0.1) + e^{\pi}$?
$x_4 = 978.26$

Ex: What is $x + y + z = 1$? Plane



To parametrize $z = 1 - x - y$

$$\vec{r}(x,y) = (x, y, 1-x-y) = x(1,0,-1) + y(0,1,-1) + (0,0,1)$$

Ex) Solve
$$\begin{cases} x_1 - x_2 + 2x_3 + 3x_4 = 2 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 - x_3 - 3x_4 = 7 \end{cases}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -3 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & -1 & 2 & 3 & 2 \\ 0 & \boxed{3} & -3 & -6 & -3 \\ 0 & 3 & -3 & -6 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} \boxed{1} & -1 & 2 & 3 & 2 \\ 0 & \boxed{1} & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right]$$

NO SOLUTIONS

$$\begin{cases} x_1 - x_2 + 2x_3 + 3x_4 = 2 \\ x_2 - x_3 - 2x_4 = -1 \\ 0 = 8 \end{cases}$$

Echelon Form (Upper Triangular) vs. Reduced Echelon Form

Echelon Form (Upper Triangular):

1. leading entries = pivots move to right
2. Entries below pivots $= 0$
3. All zero rows at bottom

Reduced Echelon Form:

1, 2, 3

4. All pivots $= 1$
5. Entries above pivot $= 0$.

Ex: Decide which of the following are in Echelon form and which are in Reduced Echelon form.

a) $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Echelon Form: b, c, d, f, g

Reduced Echelon Form: c, g