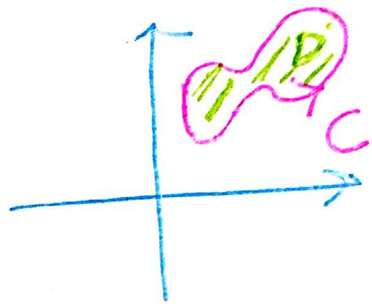
§13.5 Curl and Divergence (Contd)

What is curl $\vec{F}$? What div $\vec{F}$

Answer: Uses Green's Theorem



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$

1. Curl Form of Green's

$$\oint_C \underbrace{\vec{F}\cdot\hat{T}}_{F^{\parallel}}\, \underbrace{ds}_{d\vec{r}} = \iint_D \text{curl}\vec{F}\cdot\hat{k}\; dA$$

work = average $F^{\parallel}$ · length
‖
circulation

2. Divergence Form of Green's

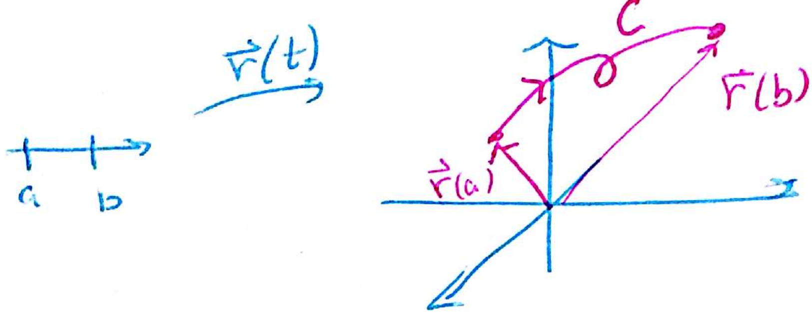
$$\oint_C \underbrace{\vec{F}\cdot\hat{n}}_{\text{normal component}}\, ds = \iint_D \text{div}\,\vec{F}\; dA$$

Flux = average $F^{\perp}$ · length

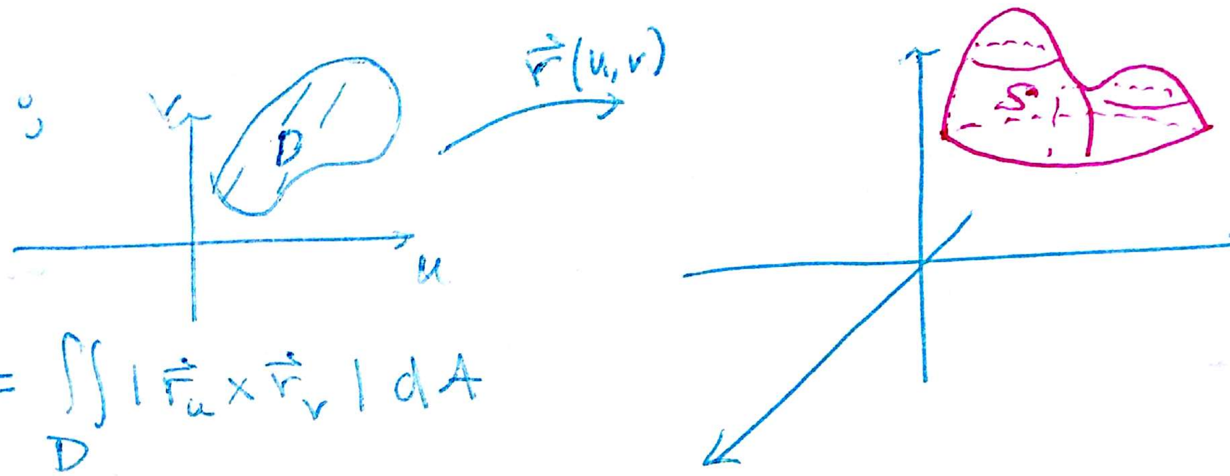
§13.6 Parametric Surfaces and their Areas

Curves (Review):

$$L(C) = \int_a^b |\vec{r}\,'(t)|\,dt$$



Surfaces:



$$\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA$$

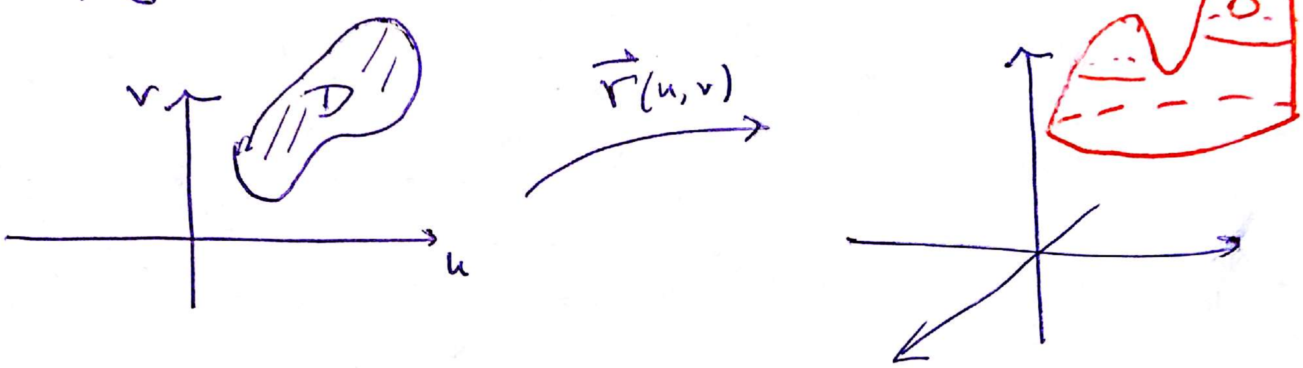
memorize surface parametrizations: 1 spheres, 2 cylind

3 Graphs $z = f(x,y)$

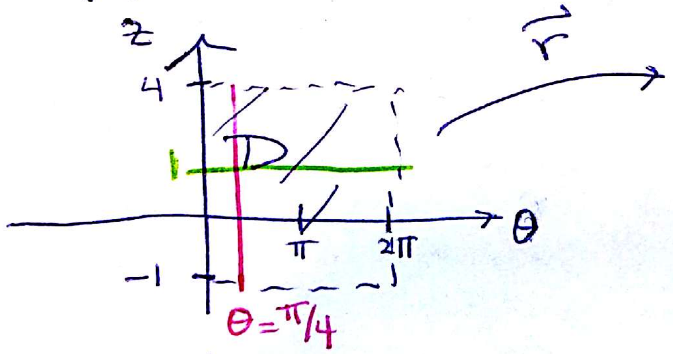
§ 13.6 Parametric Surfaces and Their Area.

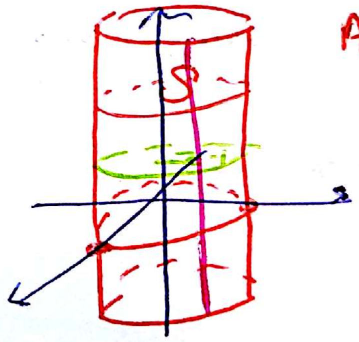
$$A(S) = \iint_S 1 \cdot dS = \iint_D \underbrace{|\vec{r}_u \times \vec{r}_v| \, du\,dv}_{dS}$$

$A(S)$: Area of S



Ex 1) Use the area formula to compute the area of the cylinder of radius 3 centered on z-axis from $z=-1$ to $z=4$





Area $= 2\pi r h$
$= 30\pi$

$\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$

$\vec{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$

$\vec{r}_z = \langle 0, 0, 1 \rangle$

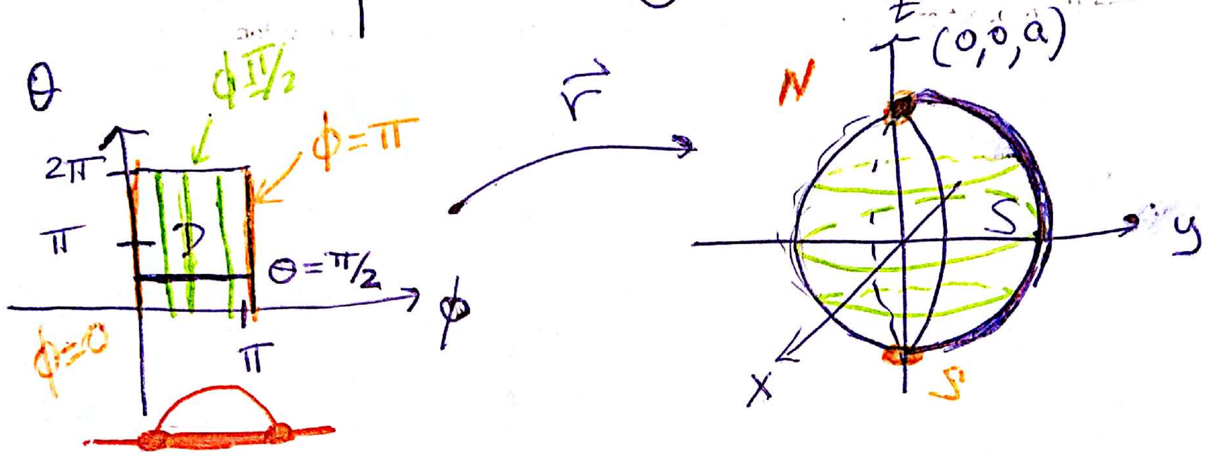
$\vec{r}_\theta \times \vec{r}_z = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$

$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{9\cos^2 + 9\sin^2} = 3$

$$A(S) = \iint_S dS = \iint_D \underbrace{|\vec{r}_u \times \vec{r}_v|}_{3} \, du\,dv = \iint_D 3\,du\,dv = 3 \cdot \text{Area}(D)$$
$$= 3 \cdot 10\pi = 30\pi$$

Scanned by CamScanner

Ex 2) Use surface area formula to compute the area of the sphere $x^2+y^2+z^2=a^2$



$$\vec{r}(\phi,\theta) = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle$$

$$0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi$$

$$\vec{r}_\phi = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0\rangle$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\cos\phi\sin\phi\rangle$$

$$= a\sin\phi\,\langle \underbrace{a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi}_{\vec{r}(\phi,\theta)}\rangle$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = a^2\sin\phi \quad \text{so} \quad dS = |\vec{r}_\phi \times \vec{r}_\theta|\,d\phi\,d\theta$$

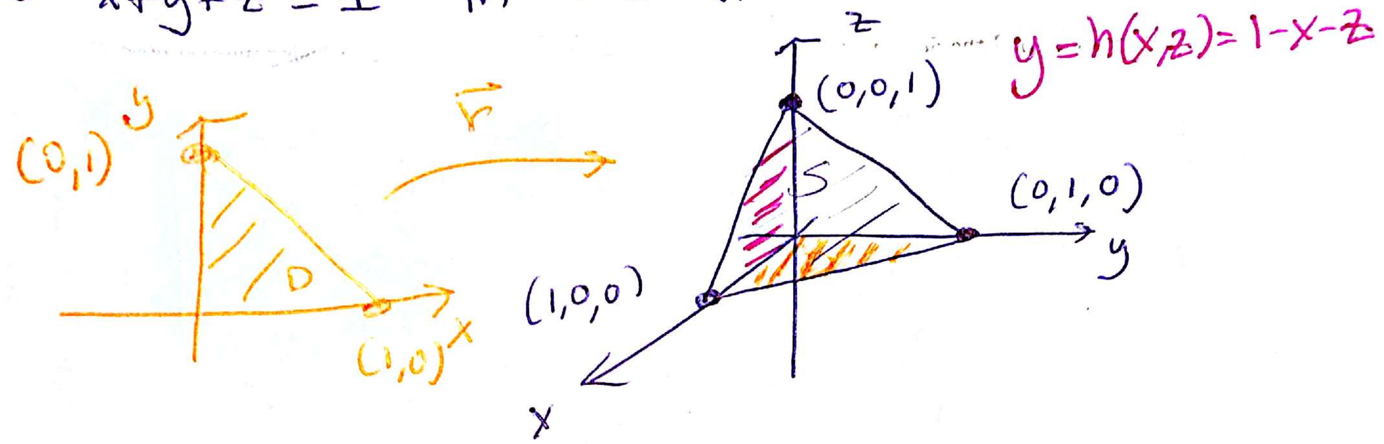
$$\boxed{dS = a^2\sin\phi\,d\phi\,d\theta}$$

$$A(S) = \iint_S dS = \iint_D^{\phi=\pi}_{\phi=0} a^2\sin\phi\,d\phi\,d\theta = \int_{\theta=0}^{\theta=2\pi} -a^2\cos\phi\,\Big|_0^\pi\,d\theta$$

$$= 2\pi\left[-a^2\cos\pi + a^2\cos 0\right]$$

$$= 4\pi a^2$$

Ex3 (Graphs) Compute the area of the plane $x+y+z=1$ in the first octant.



Graph $z=f(x,y)=1-x-y$

To parametrize graphs use shadows.

$$\vec{r}(x,y) = \langle x, y, \underset{z}{f(x,y)} \rangle$$

$$\vec{r}_x = \langle 1, 0, f_x \rangle \qquad \boxed{\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle}$$

$$\vec{r}_y = \langle 0, 1, f_y \rangle$$

$$\boxed{dS = |\vec{r}_x \times \vec{r}_y| \overset{dA}{=} \sqrt{f_x^2+f_y^2+1}\, dx\,dy}$$

In our example $f_x=-1$, $f_y=-1$

$dS = \sqrt{3}\,dx\,dy$ so $A(S) = \iint_D \sqrt{3}\,dx\,dy$

$= \sqrt{3}\,\text{Area}(D) = \frac{\sqrt{3}}{2}$

Integrate $= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \sqrt{3}\,dy\,dx = \frac{\sqrt{3}}{2}$