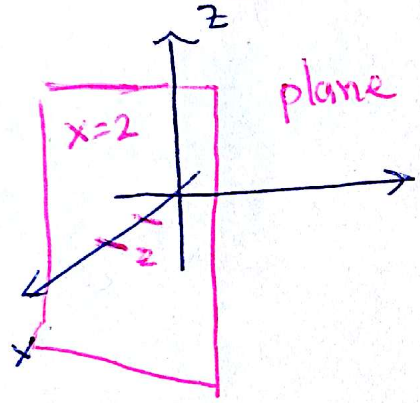
2/29 Spherical Coordinates (§12.7) Review for Exam 2. (Exam Topics: §13.5 — §13.9)

I. Rectangular coordinates x, y, z $dV = dx\,dy\,dz$

Describe geometrically the solutions to $x=2$.

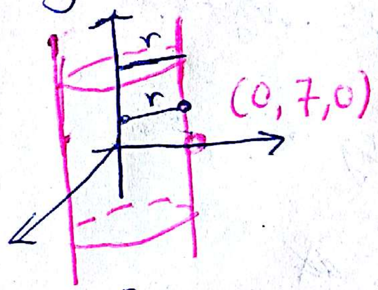


II. Cylindrical Coordinates

$\underset{\text{dist to z-axis}}{r}, \theta, z$ $dV = r\,dz\,dr\,d\theta$

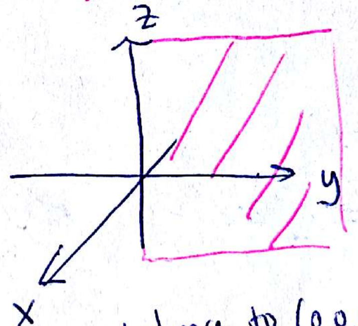
Describe geometrically the solutions

$r = 7$



cylinder centered on z-axis

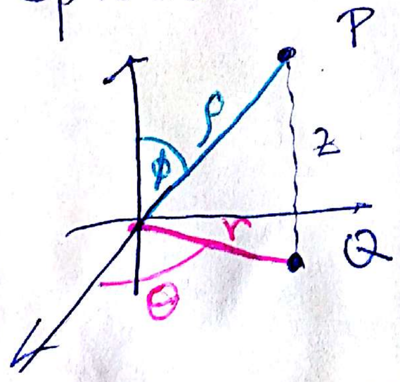
$\theta = \frac{\pi}{2}$



½ plane $x=0$

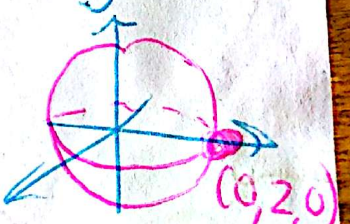
$y \geq 0$

III. Spherical coordinates: $\overset{\text{distance to }(0,0,0)}{\rho}, \underset{N/S}{\phi}, \underset{E/W}{\theta}$



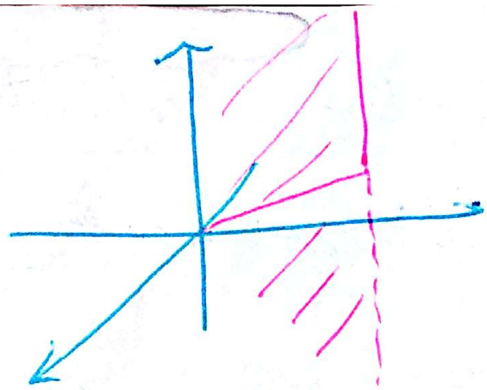
Describe Geometrically the solutions 1. $\rho = 2$

ANS: sphere of radius 2 centered (0,0,0)



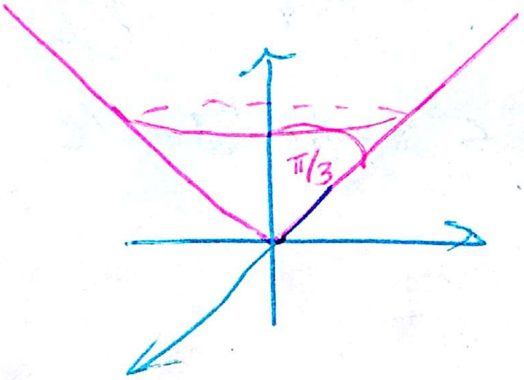
Scanned by CamScanner

2. $\theta = \frac{3\pi}{4}$



$\frac{1}{2}$ plane 135° from x-axis

3. $\phi = \pi/3$



ANSW: Cone centered on z-axis.

How to integrate in spherical coordinates?

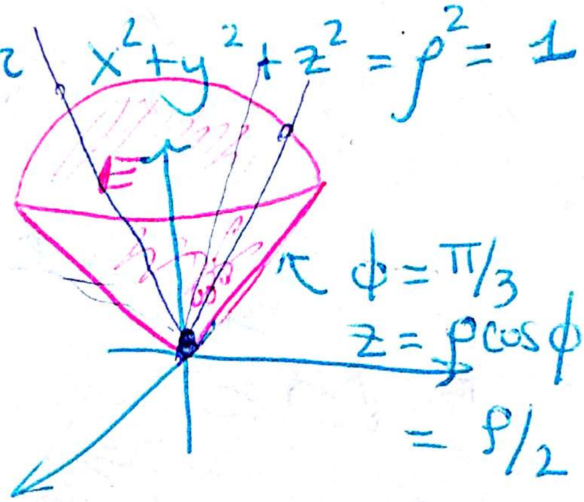
project shadow onto xy plane

$$\int_{\theta}\int_{\phi}\int_{\rho=\rho_1}^{\rho=\rho_2} f(\rho,\phi,\theta)\, dV$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

DO NOT CONFUSE WITH $\boxed{dS = a^2 \sin\phi \, d\phi \, d\theta}$

Ex) Find the volume of the ice cream cone E cut from the sphere $x^2+y^2+z^2=\rho^2=1$ and the cone $\phi=\pi/3$



$$\text{Volume} = \iiint_E 1\, dV$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/3}\int_{\rho=0}^{\rho=1} 1\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\int_{\phi=0}^{\phi=\pi/3}\frac{1}{3}\sin\phi\, d\phi = \frac{2\pi}{3}\left(-\cos\phi\right)\Big|_{\phi=0}^{\phi=\pi/3}$$

$$= \frac{\pi}{3}$$

Can we do this in cylindricals?

$$\text{Volume} = \iint\int_{z=\frac{\sqrt{x^2+y^2+z^2}}{2}}^{z=\sqrt{1-x^2-y^2}}$$

Scanned by CamScanner

Review   Math 39200 Fall 2005 Final

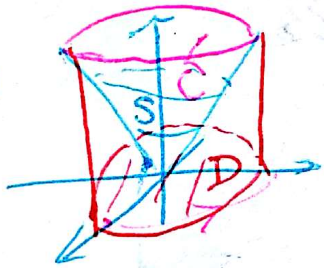
$\boxed{x^2+y^2=3^2}$

#9) C = intersection of cone $x^2+y^2=z^2$ and plane $z=3$. $\vec{F}(x,y,z)=y\hat{i}+z\hat{j}-x\hat{k}$.
Let C be oriented from above clockwise
Evaluate $\oint_C \vec{F}\cdot d\vec{r}$

a) directly   b) as a double integral.

SOLUTION

a)



C: $x=3\cos t$, $y=3\sin t$, $z=3$

$$\oint_C \vec{F}\cdot d\vec{r}=\oint_C y\,dx+z\,dy-x\,dz$$

$$=-\int_{t=0}^{t=2\pi}\left(-3^2\sin^2 t+\underbrace{3^2\cancel{\cos}t}_{0}-0\right)dt=3^2\cdot\frac{t}{2}\Big|_0^{2\pi}=9\pi$$

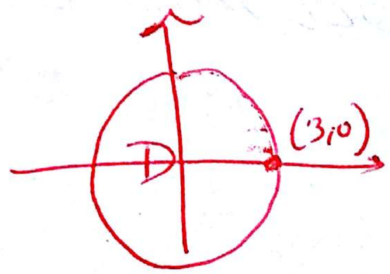
b) $\oint_C \vec{F}\cdot d\vec{r}=\iint_S \nabla\times\vec{F}\cdot d\vec{S}$ $z=\sqrt{x^2+y^2}$

$$\text{curl}\,\vec{F}=\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & z & -x \end{vmatrix}=\langle -1,1,-1\rangle \qquad d\vec{S}=\langle z_x, z_y, -1\rangle\,dx\,dy$$

$$z_x=\frac{x}{\sqrt{x^2+y^2}} \qquad z_y=\frac{y}{\sqrt{x^2+y^2}}$$

Scanned by CamScanner

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \left(\underbrace{\frac{-x}{\sqrt{x^2+y^2}}}_{0} + \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{0} + 1 \right) dx\,dy$$



$$= \iint_D 1\,dx\,dy = \text{Area}(D)$$

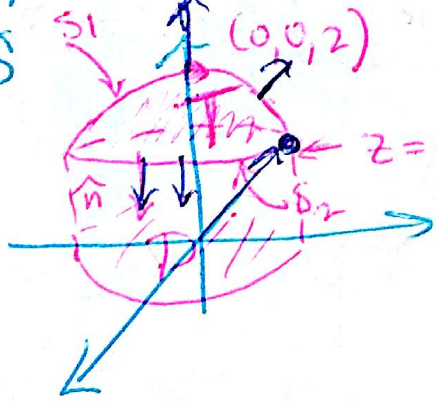
$$= 9\pi$$

~~Ex)~~

#10 $T = \begin{cases} x^2+y^2+z^2 \le 4 \\ z \geq 1 \end{cases}$ Let S be the boundary of T. $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

Use outward pointing normal to compute

$\oiint_S \vec{F}\cdot d\vec{S}$ a) directly b) as a triple integral



$S = S_1 + S_2$

a) $$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \langle x,y,z\rangle \cdot \frac{\langle x,y,z\rangle}{2}\, dS$$

$$= \iint_{S_1} 2\,dS = 2\cdot \text{Area}(S_1)$$

$$\text{Area}(S_1) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/3} 1\cdot 2^2 \sin\phi\, d\phi\, d\theta$$

$1 = z = 2\cos\phi$

$1/2 = \cos\phi$