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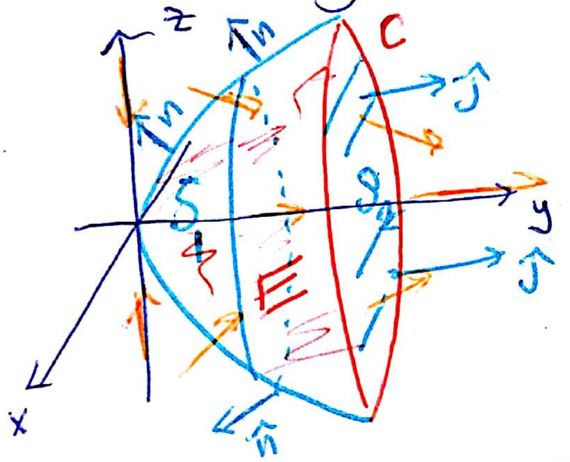
Triple Integrals (Review)

(§12.5, §12.6, §12.7)

Ex (Divergence Theorem Trick) Compute the flux of $\vec{F}(x,y,z) = y\hat{j} - z\hat{k}$ through the surface S: the paraboloid $y = x^2+z^2$ $0 \le y \le 1$



Goal: Comple

$$\text{flux} = \iint \vec{F} \cdot d\vec{S}$$

$$S = S_1 + S_2$$

$$= \iiint_E \operatorname{div} \vec{F} \, dV$$

$$\rightarrow \underbrace{\iint_{S_1} \vec{F} \cdot d\vec{S}}_{\text{Original Quest: Hard}} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = -\pi$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \vec{j}\,dx\,dz = \iint_{S_2} 1 \, dS = \text{Area}(S_2) = \pi$$

$$\iiint_E \operatorname{div} \vec{F} \, dV = 0$$

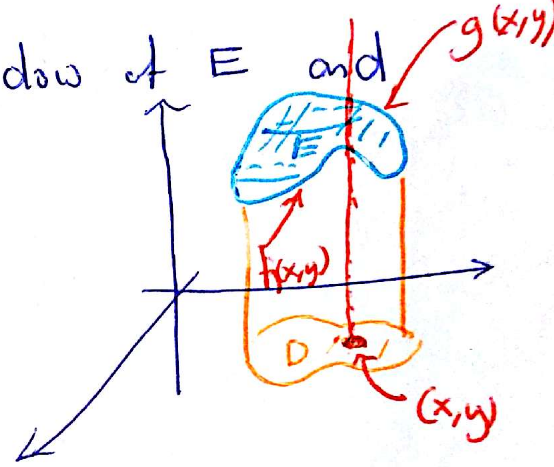
$\rightarrow$

Triple Integrals $\iiint_E f(x,y,z)\,dV$

Goal: Find dV (easy) and find the boundaries (hard)

I. Rectangular and Cylindrical. Integrate first with respect to z.

(A) Hold x, y fixed in shadow of E and let z increase + decrease



$$\iiint_E f(x,y,z)\,dV$$

$$= \iint_D \int_{z=h(x,y)}^{z=g(x,y)} f(x,y,z)\,dz\,dA$$

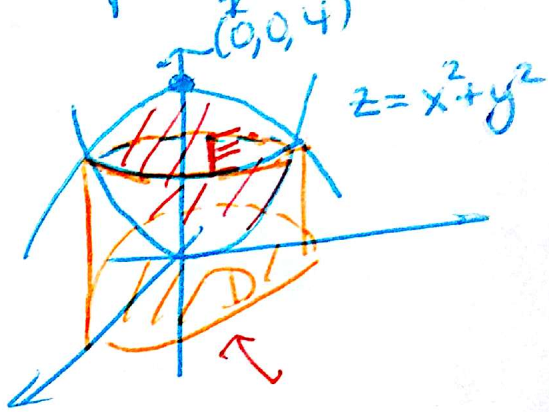
(B) Integrate over D

(i) rectangular coordinates $\displaystyle\int_{x}^{x}\int_{y=}^{y=}\int_{z=h(x,y)}^{z=g(x,y)} f(x,y,z)\,\underbrace{dz\,dy\,dx}_{dV}$

ii) in polars $\displaystyle\int_{\theta}^{\theta}\int_{r}^{r}\int_{z=h(x,y)}^{z=g(x,y)} f(x,y,z)\,\underbrace{dz\,r\,dr\,d\theta}_{dV}$

Ex) Evaluate $\iiint_E (z-1)\, dV$ when E is the solid lying between the paraboloids

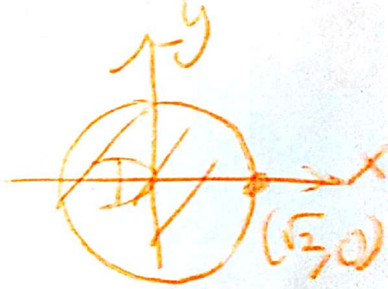
$z = x^2+y^2$ and $z = 4-x^2-y^2$



solution

$$\iint_D \int_{z=x^2+y^2}^{z=4-x^2-y^2} (z-1)\, dz\, dA$$

Use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$



To find D set $x^2+y^2 = z = 4-x^2-y^2$

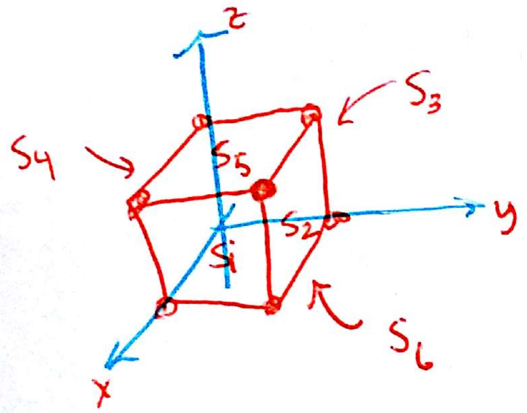
$x^2+y^2 = 2$ so $\boxed{D \quad x^2+y^2 \le (\sqrt{2})^2}$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \int_{z=r^2}^{z=4-r^2} (z-1)\, dz \cdot r\, dr\, d\theta$$

$$= 2\pi \int_{r=0}^{r=\sqrt{2}} \frac{z^2}{2} - z \Big|_{z=r^2}^{z=4-r^2} r\, dr = 2\pi \int_{r=0}^{r=\sqrt{2}} 8r - 4r^2 + \cancel{\frac{r^4}{2}} - \cancel{\frac{r^4}{2}} - 4 + 2r^2$$

$$= 2\pi \int_0^{\sqrt{2}} 4r - 2r^3\, dr = 2\pi \left[2r^2 - \frac{r^4}{2}\right]\Big|_0^{\sqrt{2}}$$
$$= 4\pi$$

Ex) Verify the divergence theorem $\vec{F} = \langle x, y, z \rangle$ and S is the surface of the cube with diagonal $(0,0,0)$ and $(1,1,1)$.



div theorem $\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV}$

A) Compute flux directly as a double integral

$d\vec{S}_1 = \hat{i}\, dy\,dz$, $d\vec{S}_2 = \hat{j}\, dx\,dz$, $d\vec{S}_3 = -\hat{i}\, dy\,dz$

$d\vec{S}_4 = -\hat{j}\, dx\,dz$, $d\vec{S}_5 = k\, dx\,dy$, $d\vec{S}_6 = -\hat{k}\, dx\,dy$

$\iint_{S_1} \underbrace{\vec{F}\cdot d\vec{S}_1}_{1} = 1$, $\iint_{S_2} \underbrace{\vec{F}\cdot d\vec{S}_2}_{=1} = 1$, $\iint_{S_3} \underbrace{\vec{F}\cdot d\vec{S}_3}_{=0} = 0$

$\iint_{S_4} \underbrace{\vec{F}\cdot d\vec{S}_4}_{=0} = 0$, $\iint_{S_5} \underbrace{\vec{F}\cdot d\vec{S}_5}_{=1} = 1$, $\iint_{S_6} \vec{F}\cdot d\vec{S}_6 = 0$

$\rightarrow$ so $\iint_S \vec{F}\cdot d\vec{S} = 3$

B) Compute flux using div theorem

$$\text{flux} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3\, dz\,dy\,dx = 3 \cdot \operatorname{Vol}(E)$$
$$= 3 \cdot 1$$

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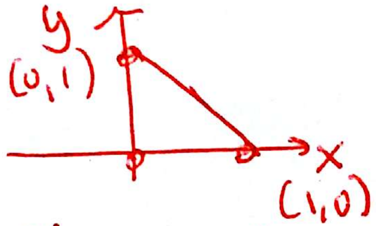
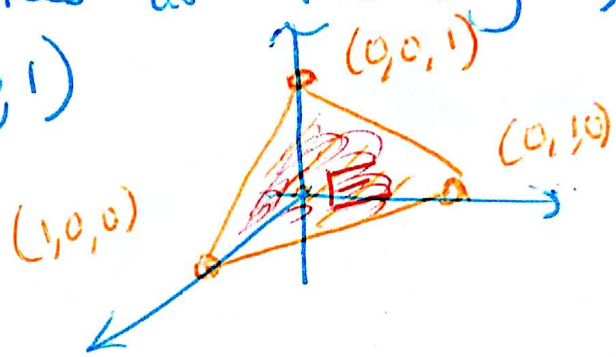
Ex) Use the divergence theorem to evaluate the flux of $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the tetrahedron with vertices at the origin, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$





$$\text{flux} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{1} \cdot dV = \operatorname{vol}(E) ??$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 1\, dz\, dy\, dx$$

$$= \int_0^1\int_0^{1-x} (1-x-y)\, dy\, dx = \int_0^1 y - xy - \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 \underbrace{(1-x) - x(1-x) - \frac{(1-x)^2}{2}}_{\frac{1}{2} - x + \frac{x^2}{2}} dx = \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\Big|_0^1 = \frac{1}{6}$$