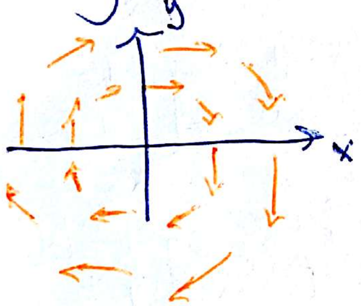
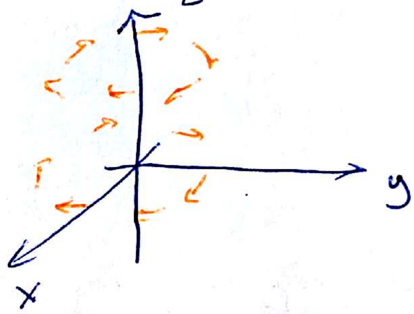
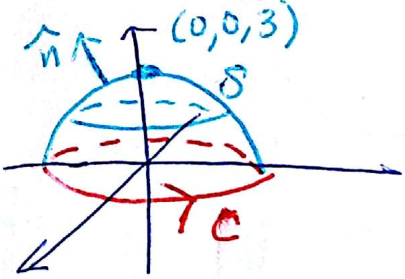
3/22 Stokes's Theorem (§13.8) + Divergence Theorem (§13.9)

Ex) Evaluate both sides of Stokes's Theorem for the hemisphere $S: x^2+y^2+z^2=9,\ z \geq 0$ (normal $\hat{n}$ up) its boundary circle $C: x^2+y^2=9,\ z=0$ and $\vec{F} = y\hat{i} - x\hat{j}$. So compute $\oint_C \vec{F}\cdot d\vec{r}$ in two ways

A) directly as a line integral and B) as a double integral using Stokes's Theorem. Sketch $\vec{F}$ for practice







SOLUTION

A) $x = 3\cos t,\ y = 3\sin t,\ z=0$
$dx = -3\sin t\,dt,\ dy = 3\cos t\,dt,\ dz = 0$

$$\text{Work (Circulation)} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C y\,dx - x\,dy = \int_0^{2\pi} -9\,dt = -18\pi$$

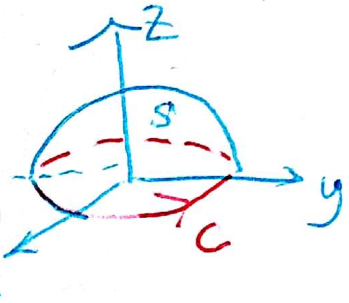
B) $\iint_S \nabla\times\vec{F}\cdot d\vec{S}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = \langle 0,0,-2\rangle$$

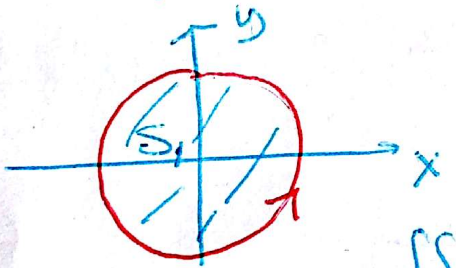
$d\vec{S} = \frac{\langle x,y,z\rangle}{3}\, dS$ $\qquad dS = 3^2 \sin\phi \, d\phi \, d\theta$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S -\frac{2z}{3}\, dS = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} -\frac{2\cdot 3\cos\phi}{3}\cdot 3^2\sin\phi \, d\phi\, d\theta$$

$$= -2\pi\cdot 2\, 3^2\, \frac{\sin^2\phi}{2}\Bigg|_{\phi=0}^{\phi=\pi/2} = -18\pi$$



Cheat: Pick an easier surface S_1



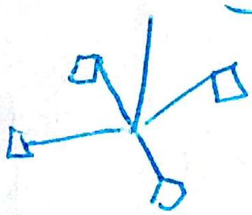
$$\iint_{S_1} \underbrace{\nabla\times\vec{F}}_{\langle 0,0,-2\rangle}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = -18\pi$$

S_1: $\hat{n} = \hat{k}$

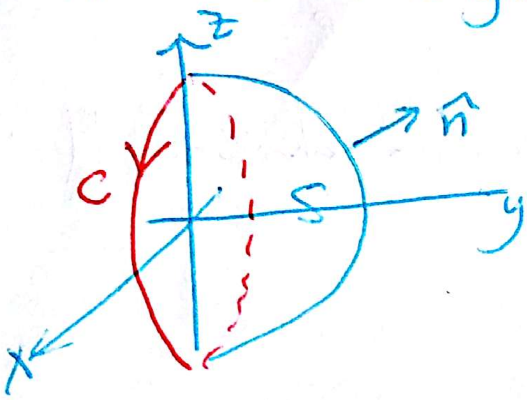
$$= \iint_{S_1} \langle 0,0,-2\rangle\cdot\hat{k}\, dA = -2\,\text{Area}(S_1) = -2\pi\, 3^2$$

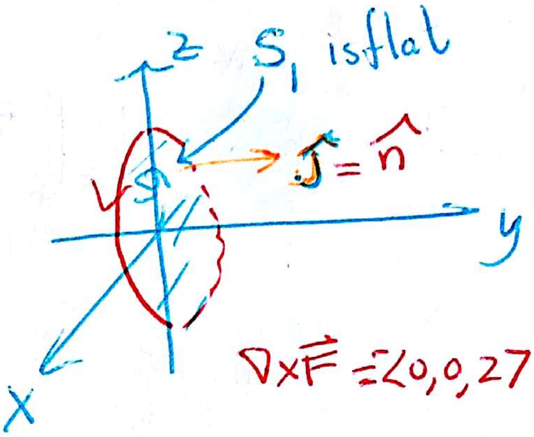
Physical Interpretation of Curl: Use Stokes's — the flux of curl through S = circulation (work) of $\vec{F}$ along boundary.

Place a paddle wheel in field



Ex) Same $\vec{F} = y\hat{\imath} - x\hat{\jmath}$ but S = right hemisphere





$\nabla \times \vec{F} = \langle 0,0,2 \rangle$

flux of the curl $= \iint_{S_1} \underbrace{\nabla \times \vec{F} \cdot \hat{n}}_{\overset{\parallel}{0}} \, dS = 0$

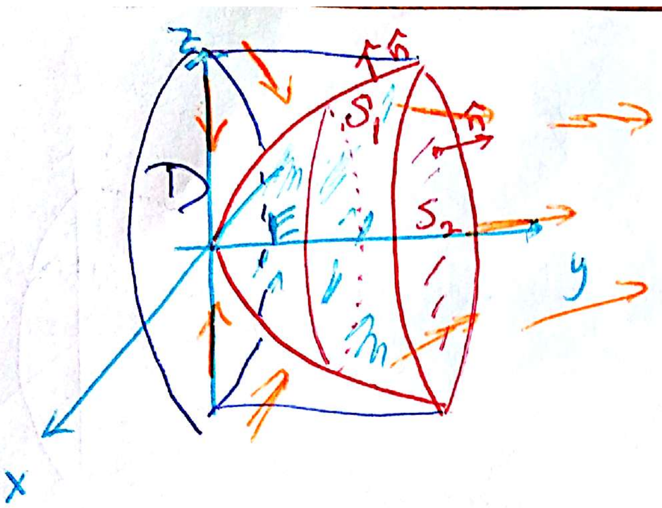
$0 = \langle 0,0,-2 \rangle \cdot \langle 0,1,0 \rangle$

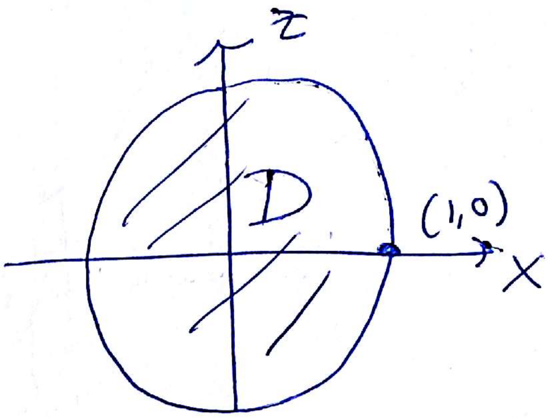
Divergence Theorem $\quad \oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$

Ex) Verify the divergence theorem when $\vec{F}(x,y,z) = y\hat{\jmath} - z\hat{k}$ and E is the volume bounded by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$, $y = 1$. i.e. Evaluate $\oiint_S \vec{F} \cdot d\vec{S}$

A) directly as a double integral

B) as a triple integral using the divergence theorem.





A) S_1 is a graph $\vec{r}(x,z) = \langle x, x^2+z^2, z \rangle$.

$$d\vec{S} = \pm \langle -y_x, 1, -y_z \rangle \, dx\,dz$$

$$d\vec{S} = \pm \langle -2x, 1, -2z \rangle \, dx\,dz = \langle 2x, -1, 2z \rangle \, dx\,dz$$

$$\vec{F} = \langle 0, y, -z \rangle$$

$$\text{flux} = \iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \langle 0, y, -z \rangle \cdot \langle 2x, -1, 2z \rangle \, dx\,dz$$

$$= \iint_{S_1} -y - 2z^2 \, dx\,dz = \iint_{S_1} (-x^2 - 3z^2) \, dx\,dz$$

POLARS
$x = r\cos\theta$
$z = r\sin\theta$

$$= \iint_{D} \int_{r=0}^{r=1} (-r^2 - 2r^2\sin^2\theta) \, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} -\frac{1}{4}\left(1 + (1 + \underbrace{\cancel{\cos 2\theta}}_{0})\right) d\theta = -\pi$$

S_2: $\hat{n} = \hat{j}$ $\quad dS = dx\,dz$

$$\text{flux} = \iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \iint_{S_2} \langle 0, +y, -z \rangle \cdot \langle 0, 1, 0 \rangle\, dx\,dy$$

$$= \iint_{S_2} \underbrace{+y}_{=1}\, dx\,dy = \text{Area}(S_2) = \pi \cdot 1^2 = \pi$$

Thus $$\oiint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

$$= -\pi + \pi = 0.$$

B) Evaluate $\iiint_E \text{div}\, \vec{F}\, dV = 0$

$$\text{div}\, \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-z) = 0 + 1 - 1 = 0$$

CONCLUSION: In this case it was much easier to compute the triple integral

$$\underbrace{\oiint_S \vec{F} \cdot d\vec{S}}_{= \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}} = \underbrace{\iiint_E \text{div}\, \vec{F}\, dV}_{=0}$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = -\pi \qquad \iint_{S_2} \vec{F} \cdot d\vec{S} = \pi$$