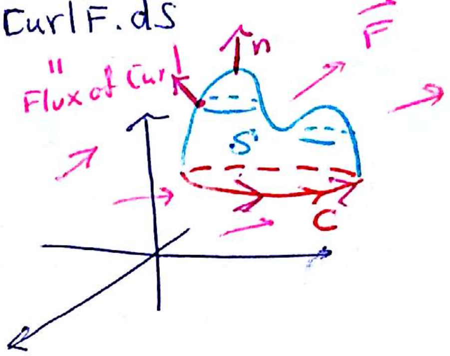
3/20

INTEGRAL THEOREMS

INTEGRAL OF BOUNDARIES = INTEGRAL OF DERIVATIVE ON REGION INSIDE

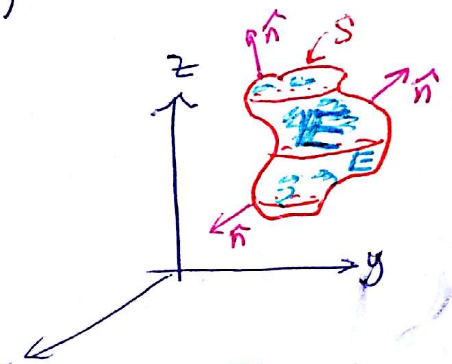
I. Stokes's Theorem (§13.8)

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{=\text{ WORK}} = \underbrace{\iint_S \operatorname{Curl}\vec{F}\cdot d\vec{S}}_{=\text{ Flux of Curl}}$$



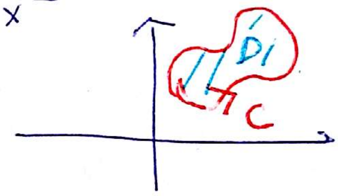
II Divergence Theorem (§13.9)

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$



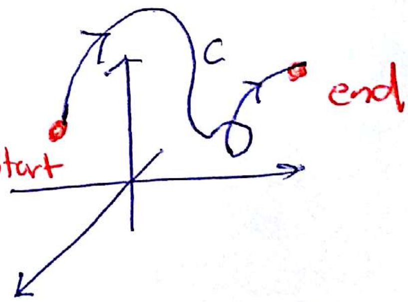
III. Green's Theorem (§13.4)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$



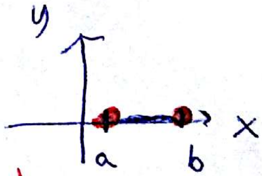
IV. FTLI (§13.3)

$$f(\text{end}) - f(\text{start}) = \int_C \nabla f\cdot d\vec{r}$$



V FTC (§4.4)

$$f(b) - f(a) = \int_a^b f'(x)\, dx$$

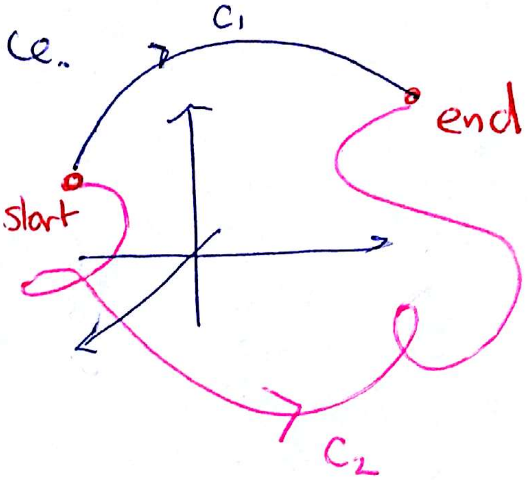


Integral on boundary = integral of derivative on inside region.

VI

$$\boxed{\int_{\partial C} \omega = \int_C d\omega}$$

FTLI and path independence.

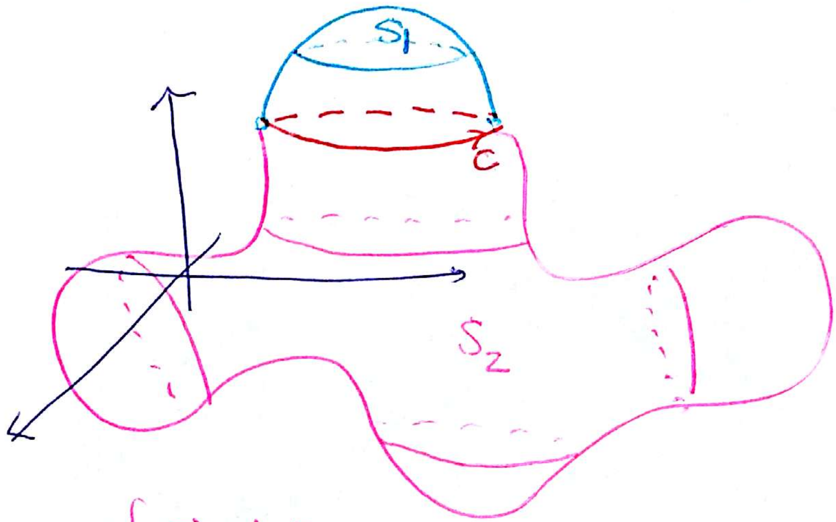


$$\int_{C_1} \nabla f \cdot d\vec{r}$$

$$\parallel \quad = f(\text{end}) - f(\text{start})$$

$$\int_{C_2} \nabla f \cdot d\vec{r}$$

Stokes's



$$\iint_{S_1} \nabla \times \vec{F} \cdot d\vec{S}$$

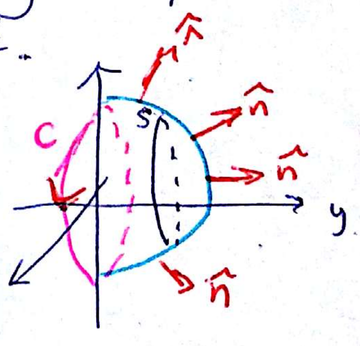
$$\parallel$$

$$\iint_{S_2} \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

Remark 1) Orientations must be compatible for Stokes's, the Divergence and Green's.

Ex) Verify Stokes's Theorem when $S = \frac{1}{2}$ sphere $x^2+y^2+z^2=9$ $y \geq 0$ (with $\hat{n}$ pointing in positive y-direction) and $\vec{F} = y\hat{i} + 2x\hat{j} + x\hat{k}$.

i.e. compute $\oint_C \vec{F} \cdot d\vec{r}$



A) Directly as a line integral

B) as a double integral using Stokes's

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A) C is parametrized $\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{k}$

$$\vec{r}(t) = \langle 3\cos t, 0, 3\sin t\rangle$$

$$\vec{r}\,'(t) = \langle -3\sin t, 0, 3\cos t\rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 0, 6\cos t, 3\cos t\rangle$$

$$\text{Work} = \oint_C \vec{F}\cdot d\vec{r} = -\int_{t=0}^{t=2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= -\int_{t=0}^{t=2\pi} 3^2 \underbrace{\cos^2 t}_{=\frac{1+\cos 2t}{2}}\,dt = -3^2\cdot\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)\Bigg|_{t=0}^{t=2\pi}$$

$$= \boxed{-9\pi}$$

B) $\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{-9\pi} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 2x & x \end{vmatrix}$$

$$\vec{r}(\phi,\theta) = \langle \underbrace{3\cos\theta\sin\phi}_{x}, \underbrace{3\sin\theta\sin\phi}_{y}, \underbrace{3\cos\phi}_{z}\rangle$$

$$\nabla\times\vec{F} = \langle 0, -1, 1\rangle$$

$$\hat{n} = \frac{\langle x,y,z\rangle}{3}, \quad \underbrace{dS = g^2\sin\phi\,d\phi\,d\theta}_{3^2\sin\phi\,d\phi\,d\theta}$$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= \iint_S \frac{-y+z}{3}\,dS = \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\pi} \frac{(-3\sin\theta\sin\phi + 3\cos\phi)}{3}\,d$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi} 3^2(-\sin\theta\sin\phi + \cos\phi)\sin\phi\,d\phi\,d\theta$$

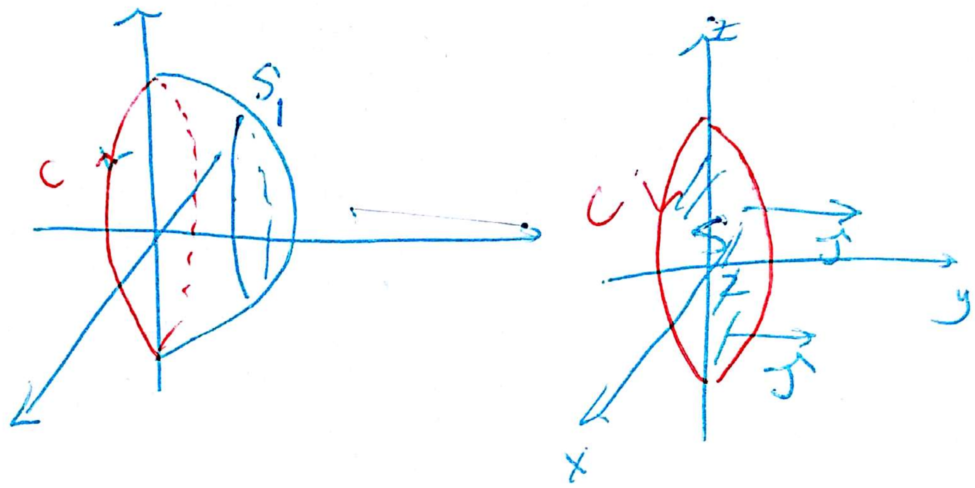
$$= 3^2\int_{\theta=0}^{\theta=2\pi}\int -\sin\theta\underbrace{\sin^2\phi}_{=\frac{1-\cos 2\phi}{2}} + \cos\phi\sin\phi\,d\phi\,d\theta$$

$$= 3^2 \int_{\theta=0}^{\theta=\pi} -\sin\theta \left(\frac{\phi}{2} - \frac{\cancel{\sin 2\phi}}{4} \right) + \frac{\cancel{\sin^2\phi}}{2} \Bigg|_{\phi=0}^{\phi=\pi} d\theta$$

$$= 3^2 \int_{\theta=0}^{\theta=\pi} -\frac{\pi}{2}\sin\theta = 3^2 \cdot \cos\theta \cdot \frac{\pi}{2} \Bigg|_{\theta=0}^{\theta=\pi}$$

$$= \boxed{-9\pi}$$

Cheat



$$\iint_{S_2} \nabla \times \vec{F} \cdot \hat{n}\, dS = \iint_{S_2} \langle 0,-1,1 \rangle \cdot \hat{j}\, dS$$

$$= \iint_{S_2} -1\, dS = -1 \cdot \text{Area}(S_2)$$

$$= -9\pi$$

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