§13.3 FTLI

$\vec{F} = \nabla g$ is gradient $\overset{\text{SAME}}{\Leftrightarrow}$ $\vec{F}$ is conservative $\overset{\text{SAME}}{\Leftrightarrow}$ $\vec{F}$ is path ind.

$\left(\oint_C \vec{F} \cdot d\vec{r} = 0\right)$

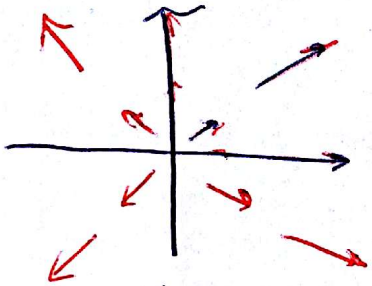
$\Downarrow$

Test $\boxed{Q_x = P_y}$ when $\vec{F} = \langle P, Q \rangle$

$\Updownarrow$ SAME

$$\boxed{\vec{O} = \text{curl}\,\vec{F} = (Q_x - P_y)\hat{k}}$$

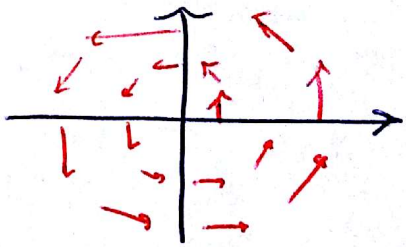
Ex: $\vec{F} = \underset{\|}{P}\hat{i} + \underset{\|}{Q}\hat{j}$



$\text{curl}\,\vec{F} = (Q_x - P_y)\hat{k} = 0\hat{k} = \vec{0}$ so $\vec{F} = \nabla f$

$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

Ex: $\vec{F} = \underset{\|}{-y}\hat{i} + \underset{\|}{x}\hat{j}$



$\text{curl}\,\vec{F} = (Q_x - P_y)\hat{k} = (1 - (-1))\hat{k} = 2\hat{k}$

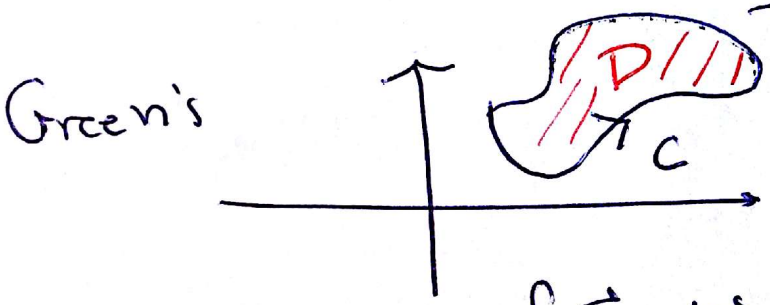
Ex) Determine if $\vec{F}$ is conservative in each of the following. If so find f.

1) $\vec{F} = xe^y \hat{i} + ye^x \hat{j}$ NOT CONSV

2) $\vec{F} = e^y \hat{i} + xe^y \hat{j}$ IS CONSV

$$f(x,y) = xe^y + C$$

§13.4 Green's Theorem



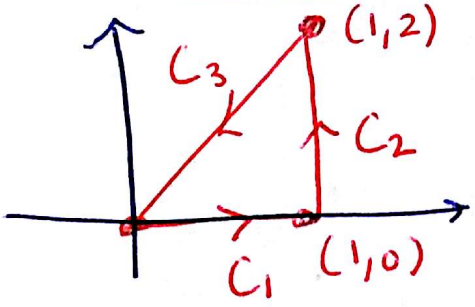
$$dA = dx\,dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA$$

Ex) Evaluate $\oint_C xy\,dx + x^2y^3\,dy$ when C is the triangle with vertices (0,0), (1,0), (1,2) in two ways

A) Directly as a line integral

B) As a double integral using Green's Thm.



$C = C_1 + C_2 + C_3$

A) Direct

C_1: $x = x$, $dx = dx$; $y = 0$, $dy = 0$; $0 \le x \le 1$

$$\int_{C_1} xy\,dx + x^2y^3\,dy = 0$$

($y = 0$, $dy = 0$)

C_2: $x = 1$, $dx = 0$; $y = y$, $dy = dy$

$$\int_{C_2} \underbrace{xy\,dx}_{0} + x^2y^3\,dy = \int_{y=0}^{y=2} y^3\,dy = \frac{y^4}{4}\Big|_0^2 = 4$$

C_3: $x = t$, $dx = dt$; $y = 2t$, $dy = 2dt$; $0 \le t \le 1$

$$\int_{C_3} xy\,dx + x^2y^3\,dy$$

$$= -\int_{t=0}^{t=1} 2t^2\,dt + \int 8t^5 \cdot 2dt$$

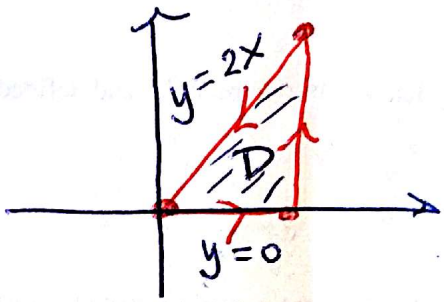
$$= -\left(\frac{2t^3}{3} + \frac{16t^6}{6}\right)\Big|_0^1$$

$$= -\left(\frac{2}{3} + \frac{8}{3}\right) = -\frac{10}{3}$$

$$\text{WORK} = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= 0 + 4 - \frac{10}{3} = \boxed{\frac{2}{3}}$$

B) Compute $\oint_C xy\,dx + x^2y^3\,dy$ using Greens



$$= \iint_D (Q_x - P_y)\,dx\,dy$$

$$= \iint_D (2y^3x - x)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} (2y^3x - x)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left(\frac{2xy^4}{4} - xy\right)\Big|_{y=0}^{y=2x} dx$$

$$= \int_{x=0}^{x=1} \left(\frac{32x^5}{4} - 2x^2\right)dx = \frac{8x^6}{6} - \frac{2x^3}{3}\Big|_0^1$$

$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$