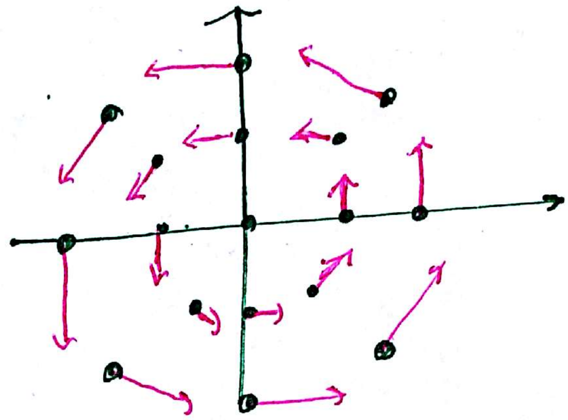
§ 13.1 (CONTD)

Ex 2) Describe the vector field $\vec{F} = -y\hat{i} + x\hat{j}$ by sketching some vectors in plane

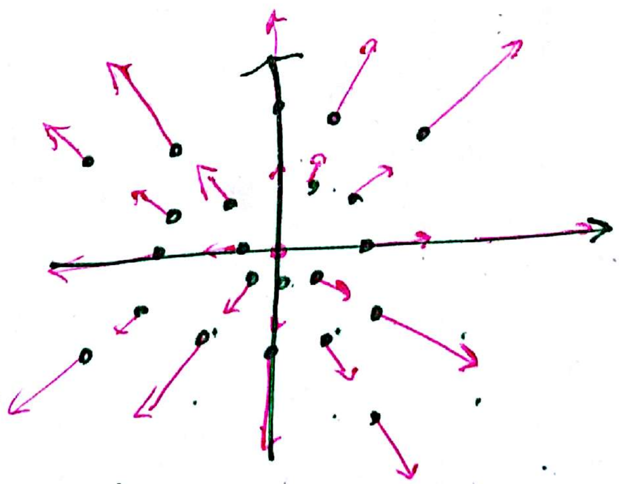
Point (x,y)	vector $\vec{F}$
(0,0)	⟨0,0⟩
(1,0)	⟨0,1⟩
(1,1)	⟨-1,1⟩
(0,1)	⟨-1,0⟩
(-1,1)	⟨1,-1⟩
(-1,0)	⟨0,-1⟩
(-1,-1)	⟨1,-1⟩



Ex 3) Describe the vector field $\vec{F} = x\hat{i} + y\hat{j}$

by sketching

Point (x,y)	vector $\vec{F}(x,y)$
(0,0)	⟨0,0⟩
(1,0)	⟨1,0⟩
(1,1)	⟨1,1⟩
(0,1)	⟨0,1⟩
(-1,1)	⟨-1,1⟩
(-1,0)	⟨-1,0⟩
(-1,-1)	⟨-1,-1⟩
(0,-1)	⟨0,-1⟩
(1,-1)	⟨1,-1⟩
(2,0)	⟨2,0⟩
(2,2)	⟨2,2⟩



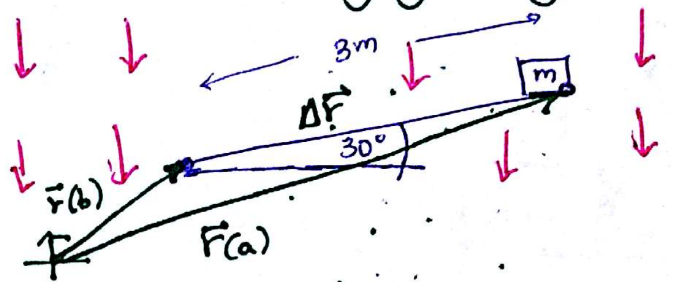
§ 13.2 Line Integrals

We often use line integrals to compute work.

Ex) Compute the work done by gravity

$m = 10\,kg$

$\vec{F} = -mg\hat{j}$



High School Way: Work = Force · distance = $\underbrace{mg\sin 30^\circ}_{F''} \cdot \underbrace{3}_{dist = \Delta s}$

Vector Way: Work = $\vec{F} \cdot \Delta\vec{r} = \langle 0, -mg \rangle \cdot 3\langle \cos 30^\circ, -\sin 30^\circ \rangle$

Calculus Way: Work = $\lim_{\Delta\vec{r}_i \to 0} \sum \vec{F}_i \cdot \Delta\vec{r}_i = \int_C \vec{F} \cdot d\vec{r}$

Each way gives the same answer in this simple case when $\vec{F} = -98\hat{j}$ is constant and C is a line segment. However most vector fields change and most curves are not straight. For those problems the calculus way is necessary.

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Line Integral $\int_C \vec{F}\cdot d\vec{r}$

1) Direct $\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

2) Geometric Way: $\int_a^b \vec{F}\cdot\hat{T}\,ds$

3) Differentials $\int_C P\,dx+Q\,dy$ when $\vec{F}=\langle P,Q\rangle$

Ex) Redo $\vec{F}=\langle -y,x\rangle$ and $\vec{r}(t)=\langle \underset{x}{\underset{\shortparallel}{t}},\underset{y}{\underset{\shortparallel}{t^2}}\rangle$ $0\le t\le 1$

using differentials.

$P(x,y)=-y$

$Q(x,y)=x$

$$\boxed{\begin{array}{ll} x=t & dx=dt \\ y=t^2 & dy=2t\,dt \end{array}}$$

$$\int_C P\,dx+Q\,dy=\int_0^1 -t^2\cdot dt+t\cdot 2t\,dt$$

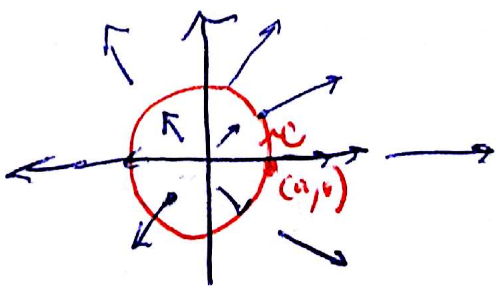
$$=\int_0^1 t^2\,dt=1/3$$

Ex: Compute $\int_C x\,dx+y\,dy$ when C is the circle of radius a centered at (0,0).

Work $=\int_C \underbrace{\vec{F}\cdot\hat{T}}_{=0}\,ds=0$

$\vec{F}(x,y)=\langle x,y\rangle$

Point	$\vec{F}$
(1,2)	$\langle 1,2\rangle$



§13.2 Line Integrals

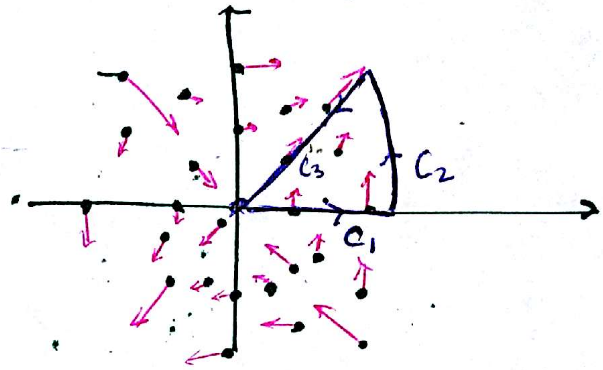
Ex) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ when

$\vec{F} = y\hat{i} + x\hat{j}$ and $C = C_1 + C_2 + C_3$

C_1 = x-axis $0 \le x \le 1$

C_2 = unit circle $0 \le \theta \le \pi/4$

C_3 = line $y = x$



C_1: Geometric $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \underbrace{\vec{F}\cdot\hat{T}}_{0}\, ds = 0$

C_2: $x = \cos t$ $dx = -\sin t\, dt$
$y = \sin t$ $dy = \cos t\, dt$
$0 \le t \le \pi/4$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} y\,dx + x\,dy = \int_{t=0}^{t=\pi/4} -\sin^2 t + \cos^2 t\, dt$$

$$= \int_{t=0}^{t=\pi/4} \cos(2t)\, dt = \frac{1}{2}$$

C_2: backwards parametrize $\begin{cases} x = x & dx = dx \\ y = x & dy = dx \end{cases}$

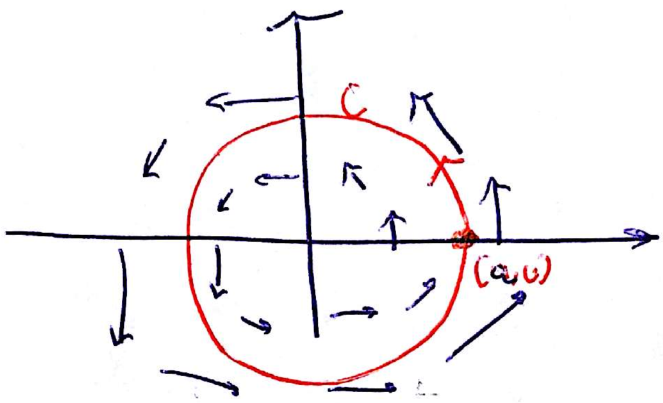
$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_3} y\,dx + x\,dy = -\int_{x=0}^{x=1/\sqrt{2}} 2x\, dx = -x^2\Big|_0^{1/\sqrt{2}} = -1/2$$

so $\int_C \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot dr + \int_{C_3} \vec{F}\cdot d\vec{r} = 0 + \frac{1}{2} - \frac{1}{2} = 0$

Example) (Closed curve nonzero work)

Compute Work $= \int_C \vec{F} \cdot d\vec{r}$ $\quad \vec{F} = \langle -y, x \rangle$

$C: x^2 + y^2 = a^2$ parametrized counter clockwise



$$\oint_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int_C -y\,dx + x\,dy = \int_{t=0}^{t=2\pi} \left(a^2 \sin^2 t + a^2 \cos^2 t\right) dt$$

$$\begin{cases} x = a\cos t & dx = -a\sin t\,dt \\ y = a\sin t & dy = a\cos t\,dt \end{cases} \qquad 0 \le t \le 2\pi$$

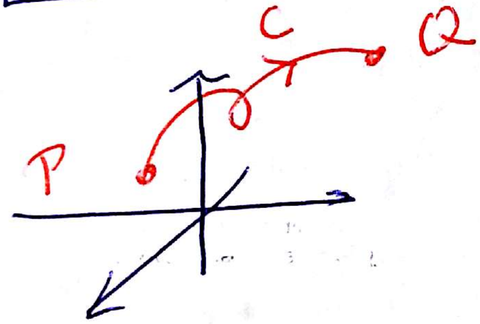
$$\text{WORK} = \int_0^{2\pi} a^2\,dt = \boxed{2\pi a^2}$$

How can we avoid computing line integrals?

Answ: If $\vec{F}$ is conservative, a gradient $\vec{F} = \nabla f$ then use the FTLI (§3.3)

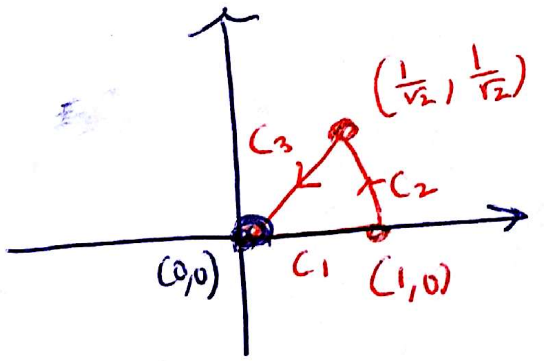
f = scalar = potential of $\vec{F}$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)}$$



$$\int_C \vec{F}\cdot d\vec{r} = f(0,0) - f(0,0) = 0$$

Ex) $\vec{F}(x,y) = \langle y, x\rangle = \nabla f$



$$\boxed{f(x,y) = xy}$$

$$\nabla f = \langle \underset{\substack{\parallel \\ y}}{f_x}, \underset{\substack{\parallel \\ x}}{f_y} \rangle$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = f(1,0) - f(0,0) = 0 - 0 = 0$$

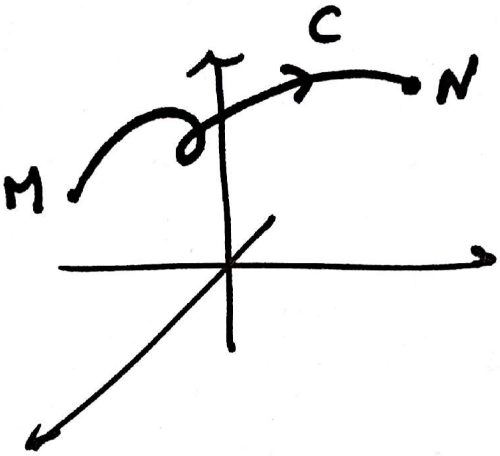
$$\int_{C_2} \vec{F}\cdot d\vec{r} = f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) - f(1,0) = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = f(0,0) - f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = 0 - \tfrac{1}{2}$$

§13.3 FTLI

If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot dr$

$= f(N) - f(M)$



1. When is $\vec{F}$ gradient?

2. If $\vec{F}$ is gradient, how to find f?

Answers.

1. $\vec{F} = \langle P, Q \rangle$

$\quad \parallel$

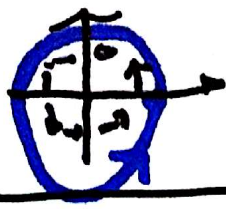
$\nabla f = \langle f_x, f_y \rangle$

$$\begin{cases} f_x = P \\ f_y = Q \end{cases}$$

$$Q_x = f_{yx} = f_{xy} = P_y$$

gradient test $\boxed{Q_x = P_y}$

Ex) $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$



Test Fails

$P(x,y) = -y$

$Q(x,y) = x$

$$\boxed{Q_x = 1 \neq -1 = P_y}$$

Ex) Find the constant a so that $\cancel{\vec{F}}$

$$\vec{F} = \underbrace{(4x^2 + axy)}_{P}\hat{\imath} + \underbrace{(3y^2 + 4x^2)}_{Q}\hat{\jmath} \quad \text{is conservative.}$$

$Q_x = P_y$ if $a = 8$. If $a = 8$, $\vec{F}$ is conservative.

Ex) (Harder) Knowing $\vec{F} = (4x^2 + 8xy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$ is conservative, find a potential function $f(x,y)$ so that $\nabla f = \vec{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\begin{cases} f_x = 4x^2 + 8xy \xrightarrow[\text{integrate}]{①} f = \int (4x^2 + 8xy)\,dx = \dfrac{4x^3}{3} + 4x^2y + g(y) \\ f_y = 3y^2 + 4x^2 \xleftarrow[\text{substitute}]{②} \end{cases}$$

② $$0 + 4x^2 + g'(y) = f_y = 3y^2 + 4x^2$$

$$g'(y) = \cancel{4}\,3y^2$$

$$g(y) = y^3$$

so $$\boxed{f(x,y) = \frac{4}{3}x^3 + 4x^2y + y^3}$$