APPROXIMATE INTEGRALS HANDOUT

A little intro will be done in class, so lets just jump into it.

Let $\displaystyle\int_a^b f(x)\,dx$ be a definite integral that we want to approximate. Let n be the number of subintervals we wish to use in the approximation (that is, how many pieces we want to cut the interval $[a,b]$ into), and $\Delta x = \dfrac{b-a}{n}$ be the length of each subinterval. Moreover, let M_n, T_n and S_n denote the Midpoint rule, Trapezoidal rule and Simpson's rule approximations with n subintervals, respectively. Then we have that,

1. The Midpoint Rule:
   $$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$
   where $\bar{x}_i =$ midpoint of $[x_{i-1}, x_i]$

2. The Trapezoidal Rule:
   $$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$
   where $x_i = a + i\Delta x$

3. Simpson's Rule:
   $$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$
   where n is even and $x_i = a + i\Delta x$

   In class, we also covered:

4. The Righthand rule:
   $\displaystyle\int_a^b f(x)\,dx \approx R_n = \Delta x\,[f(x_1) + f(x_2) + \cdots + f(x_n)]$, ($f$ is evaluated at the right end-points of the subintervals) and

5. The Lefthand rule:
   $\displaystyle\int_a^b f(x)\,dx \approx L_n = \Delta x\,[f(x_0) + f(x_2) + \cdots + f(x_{n-1})]$, ($f$ is evaluated at the left end-points of the subintervals)

   (In all of the above, $x_0 = a$ and $x_n = b$)

The above give approximations for the definite integral $\int_a^b f(x)\ dx$, but an approximation isn't very useful if we're not sure of how good it is. The difference between an approximaiton and the actual answer is called *the error* of the approximation. The errors associated with each of the above approximation rules are given by:

1. $|E_M| \leq \dfrac{K(b-a)^3}{24n^2}$. This gives a bound for the error with the midpoint rule.
2. $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$. This gives a bound for the error with the trapezoidal rule.

   For the above, K is a constant such that $|f''(x)| \leq K$ for all $a \leq x \leq b$.
3. $|E_S| \leq \dfrac{K(b-a)^5}{180n^4}$. This gives a bound for the error with Simpson's rule.

   For E_S, K is a constant such that $\left|f^{(4)}(x)\right| \leq K$ for all $a \leq x \leq b$.

Problems:

1. Approximate $\int_0^\pi \sin x\ dx$ using 4 subintervals and (a) Midpoint rule, (b) Trapezoid rule, and (c) Simpson's rule.
2. Find the errors associated with the approximations in problem 1 (in terms of n).
3. How many subintervals are needed to make sure that the error associated with the Trapezoidal rule in approximating $\int_1^2 \frac{1}{x}\ dx$ is less than $\frac{1}{100}$?
4. Set up the Midpoint approximation for $\int_1^2 \sqrt{1+x^3}\ dx$, $n = 4$.