MATH 201

FALL 2017 FINAL EXAM SOLUTIONS

(1) (a) $f(x) = \tan(x\sqrt{1+x^2})$

$$f'(x) = \sec^2(x\sqrt{1+x^2})\cdot\left(x\left(\tfrac{1}{2}\right)(1+x^2)^{-1/2}(2x) + (1+x^2)^{1/2}\right)$$
$$= \sec^2(x\sqrt{1+x^2})\cdot\left(x^2(1+x^2)^{-1/2} + (1+x^2)^{1/2}\right)$$

$$= \sec^2(x\sqrt{1+x^2})\cdot(1+x^2)^{-1/2}(x^2+1+x^2)$$
$$= \boxed{\frac{(2x^2+1)\sec^2(x\sqrt{1+x^2})}{\sqrt{1+x^2}}}$$

(b) $f(x) = \underbrace{\sin(x^3)}_{u}\underbrace{\cos^2(5x)}_{v}$

$$u' = \cos(x^3)\cdot 3x^2 \quad \& \quad v' = 2\cos(5x)\cdot(-\sin(5x))\cdot 5$$
$$= -10\sin(5x)\cos(5x)$$

So $f'(x) = \boxed{\sin(x^3)\cdot(-10\sin(5x)\cos(5x)) + \cos^2(5x)\cdot\cos(x^3)\cdot 3x^2}$

(c) $f(x) = \left(\frac{3x-4}{x^2+7}\right)^7$

$$f'(x) = 7\left(\frac{3x-4}{x^2+7}\right)^6\left(\frac{\overbrace{(x^2+7)(3)}^{3x^2+21} - \overbrace{(3x-4)(2x)}^{6x^2-8x}}{(x^2+7)^2}\right)$$

$$= \boxed{\frac{7(3x-4)^6(-3x^2+8x+21)}{(x^2+7)^8}}$$

② (a) $\int (x^2-1)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

$= \int \left(x^{5/2}+x^{3/2}-x^{1/2}-x^{-1/2}\right)dx$

$= \boxed{\frac{2}{7}x^{7/2}+\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}-2x^{1/2}+C}$

(b) $\int_0^1 \frac{x^{99}}{(7x^{100}+1)^3}dx$ 　 let $u = 7x^{100}+1$

$du = 700x^{99}\,dx$

$\Rightarrow \frac{1}{700}du = x^{99}dx$

$\frac{1}{700}\int_1^8 u^{-3}\,du = \frac{1}{700}\left.\frac{u^{-2}}{-2}\right|_1^8 = -\frac{1}{1400}\left(\frac{1}{64}-1\right)$

$= \boxed{\frac{63}{64(1400)}} = \frac{9}{64(200)}$

$= \boxed{\frac{9}{12800}}$

(c) $\int \frac{\sin x}{\cos^2 x}dx$ 　 Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$-\int u^{-2}du = -\frac{u^{-1}}{-1}+C = \frac{1}{u}+C = \boxed{\sec x + C}$

OR notice $\frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x \ldots$

③ (a) $\lim_{x\to -1} \frac{x^2+3x+2}{x^2-3x-4} = \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x-4)} = \lim_{x\to -1} \frac{x+2}{x-4} = \frac{(-1)+2}{(-1)-4}$

$= \frac{1}{-5} = \boxed{-\frac{1}{5}}$

(b) $\lim_{x\to\infty} \frac{2x^3+x}{3x^3-x^2+4}$ ← Rational function & deg (numerator) = deg (denominator)

$\Rightarrow$ limit is $\boxed{\frac{2}{3}}$ OR $\lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{3-\frac{1}{x}+\frac{4}{x^3}} = \frac{2+0}{3-0+0} = \boxed{\frac{2}{3}}$

(c) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$. By Squeeze Theorem $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

④ (a) $f(x) = \sqrt{x}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \left[\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}\right]$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$

(b) $f(1) = 1$ & $f'(1) = \frac{1}{2(1)} = \frac{1}{2} \Rightarrow$ Tangent line: $\boxed{y-1 = \frac{1}{2}(x-1)}$

⑤ (a) $F(x) = \int_0^{\tan x} \sqrt{1-t^3}\, dt.$

By FTC & Chain Rule

$$F'(x) = \boxed{\sqrt{1-\tan^3 x} \cdot \sec^2 x}$$

(b) $x^2 + 2xy - y^2 = 2$

$$\frac{d}{dx} \Rightarrow 2x + 2\left[x\frac{dy}{dx} + y\right] - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + x\frac{dy}{dx} + y - y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y = (y-x)\frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{x+y}{y-x}}$$

⑥

$$f(x) = \frac{x^3}{x^2-1} = \frac{x^3}{(x+1)(x-1)}$$

$$f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2}$$

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

(a) $(x+1)(x-1) = 0 \Rightarrow x = 1$ OR $x = -1$

Domain: $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$

(b) y-intercept: $x = 0 \Rightarrow y = f(0) = \frac{0}{-1} = 0$

x-intercept(s): $y = 0 \Rightarrow x^3 = 0 \Rightarrow x = 0$

Horizontal Asymptote(s): $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x = +\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (-x) = -\infty$

Vert. Asymptote(s): $x = 1$ & $x = -1$

(c) $f'(x) = 0 \Rightarrow x^2(x^2-3) = 0 \Rightarrow x = 0$ OR $x = \sqrt{3}$ OR $x = -\sqrt{3}$

	$x<-\sqrt{3}$	$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	x
test point	-2		-1.5		$-1/2$		$1/2$		1.5		2
x^2-3	$+$		$-$		$-$		$-$		$-$		$+$
$f'(x)$	$+$		$-$		$-$		$-$		$-$		$+$
$f(x)$	$\nearrow$		$\searrow$		$\searrow$		$\searrow$		$\searrow$		$\nearrow$

Increase: $(-\infty, -\sqrt{3})$ & $(\sqrt{3}, \infty)$

Decrease: $(-\sqrt{3}, -1), (-1, 0), (0, 1), (1, \sqrt{3})$

Critical Points: $\left(-\sqrt{3}, \frac{-3\sqrt{3}}{2}\right)$ ← Local max

$(0, 0)$

$\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$ ← Local min

(d) Concavity: Analyze $f''(x)$

$$f''(x)=0 \Rightarrow 2x(x^2+3)=0 \Rightarrow x=0$$

		-1		0		1	
test point	-2		$-1/2$		$1/2$		2
$2x$	$-$		$-$		$+$		$+$
$(x^2-1)^3$	$+$		$-$		$-$		$+$
$f''(x)$	$-$		$+$		$-$		$+$
Concavity of $f(x)$	$\cap$		$\cup$		$\cap$		$\cup$

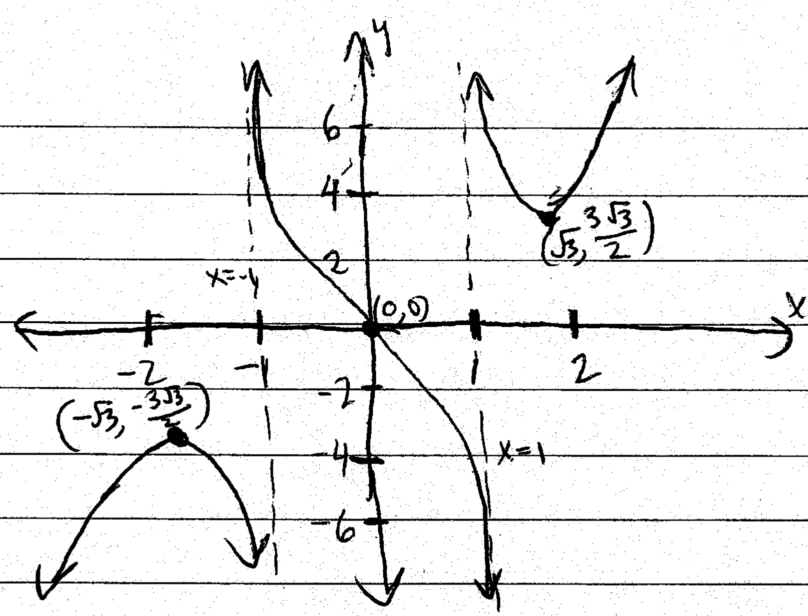
Concave Up: $(-1, 0), (1, \infty)$

Concave Down: $(-\infty, -1), (0, 1)$

Point of inflection: $(0, 0)$ ← Change in concavity

(e) See (c) & (d)

(f)



⑦ (a) $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

<u>Given:</u> $\frac{dV}{dt} = 5$

<u>Snapshot of Interest:</u> $\frac{dS}{dt} = ?$ when $r = 20$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ ← Need this at snapshot

Since $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ & $\frac{dV}{dt} = 5$, we have

$4\pi(20)^2 \frac{dr}{dt} = 5$ at the snapshot of interest

$\Rightarrow \frac{dr}{dt} = \frac{5}{4\pi(20)^2} = \frac{1}{320\pi}$ (ft/min)

So $\frac{dS}{dt} \overset{\text{at snapshot}}{=} 8\pi(20\text{ ft}) \cdot \frac{1}{320\pi}\text{ ft/min} = \boxed{\frac{1}{2}\text{ ft}^2/\text{min}}$

(b) $f'(x) = \cos x - \sin x$. If $f'(x) = 0$ $\cos x = \sin x \xrightarrow{0 \le x \le \frac{\pi}{2}} x = \frac{\pi}{4}$

<u>$x=0$:</u> $f(0) = 1$ ← absolute min $\quad x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) = 1 \quad x = \frac{\pi}{4}$: $f(\frac{\pi}{4}) = \sqrt{2}$ ← absolute max

(8) (a) $s - s_0 = v_0(t - t_0)$

$$s + 3 = 5(t - 0) = 5t$$

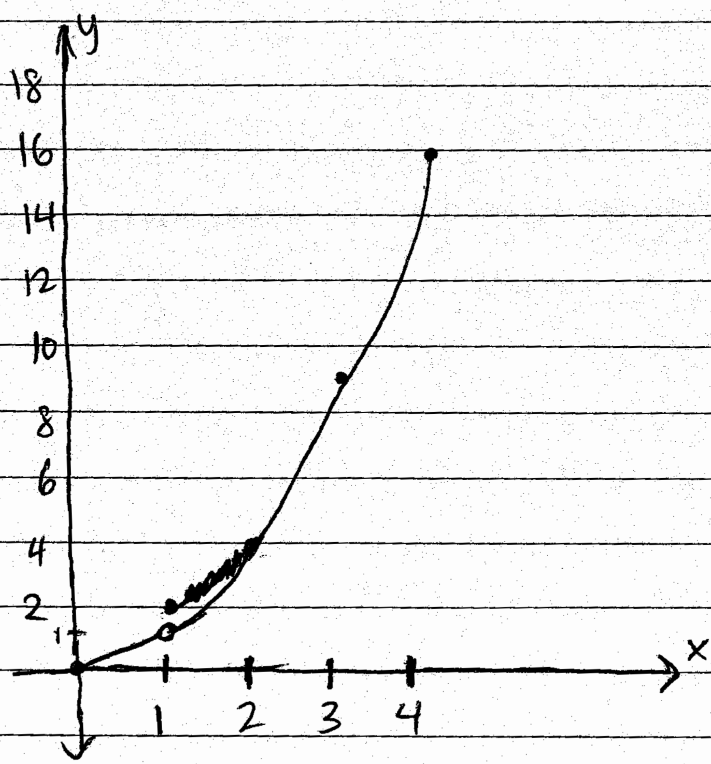
So $\boxed{L(t) = -3 + 5t}$

(b) $s(2) \approx L(2) = -3 + 5(2) = -3 + 10 = \boxed{7}$

(9) $f(x) = \begin{cases} x, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ x^2, & \text{if } x > 1 \end{cases}$

(a)

(i)

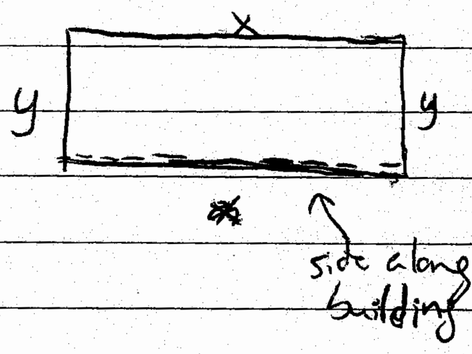


(ii) $\left.\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1} x^2 = 1 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1} x = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = \boxed{1}$

(iii) No. $\lim_{x \to 1} f(x) = 1$, but $f(1) = 2$ & $1 \neq 2$.

(b) Left Endpoints: 0, 1, 2, 3 $\quad L_4 = 1 \cdot (f(0) + f(1) + f(2) + f(3)) - $ [illegible]

(10)



Maximize $A = xy$ subject to

$x + 2y = 500$

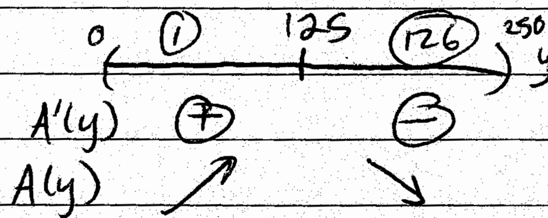
$x = 500 - 2y$

$\Rightarrow A = A(y) = (500-2y)y$

$= 500y - 2y^2, \quad 0 < y < 250$

$A'(y) = 500 - 4y$

If $A'(y) = 0$, then $500 = 4y \Rightarrow y = 125$



So setting $\boxed{y=125}$ maximizes the area of the field.

The corresponding x is $500 - 2(125) = 500 - 250 = \boxed{250}$

(11) $f: (0, +\infty) \longrightarrow \mathbb{R}$ CONTINUOUS

(a) $f(1) = f(4) = 0$

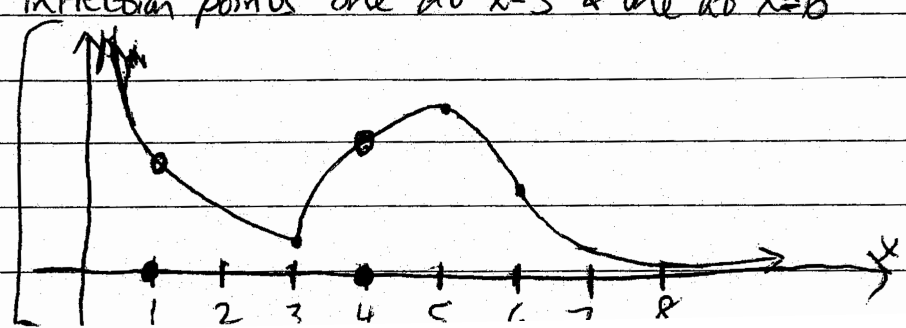
(b) $\lim_{x \to 0^+} f(x) = +\infty$

(c) $\lim_{x \to \infty} f(x) = 0$

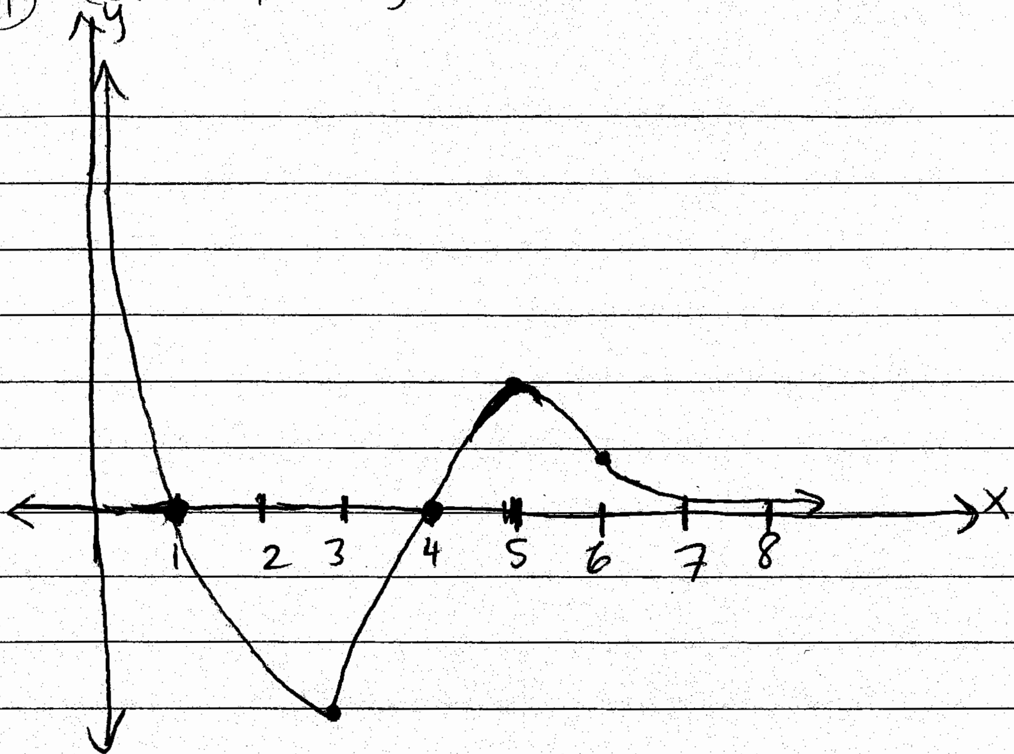
(d) f has a local min at $x=3$ and a local max at $x=5$

(e) f has two inflection points one at $x=3$ & one at $x=6$

But this is not continuous!



(11) (continued/redone)



(12) (a)(i) $\lim_{x\to\infty} \frac{\sin x \cos x}{x}$

$-1 \le \sin x \cos x \le 1 \xrightarrow{x>0} -\frac{1}{x} \le \frac{\sin x \cos x}{x} \le \frac{1}{x}$

$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{1}{x} = 0$

So by the Squeeze Theorem $\lim_{x\to\infty} \frac{\sin x\cos x}{x} = \boxed{0}$

(ii) $\lim_{x\to\infty} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to\infty} \frac{\sqrt{\frac{x+5}{x^2}} - \frac{3}{x}}{\frac{x}{x} - \frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x}+\frac{5}{x^2}} - \frac{3}{x}}{1-\frac{4}{x}}$

$= \frac{\sqrt{0+0} - 0}{1-0} = \frac{0}{1} = \boxed{0}$

OR recognize as $f'(4)$ for $f(x) = \sqrt{x+5}$

(iii) $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \lim_{x\to 4} \frac{\overbrace{(x+5)-9}^{\cancel{x-4}}}{(x-4)(\sqrt{x+5}+3)}$

$= \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$

(b) $x^7+x-1=0$

Let $f(x)=x^7+x-1$ ← polynomial (continuous everywhere)

$f(0)=-1<0$ & $f(1)=1+1-1=1>0$

By Intermediate Value Theorem, f takes the value 0 somewhere between 0 & 1.

f cannot take this value more than once because $f'(x)=7x^6+1>0$ for all x so that f is always increasing. (Once it hits zero, it cannot hit that value again.)