# 4 Exponential and Logarithmic Functions

## 4.1 Exponential Functions

### Example

Use the graph of $f(x) = 2^x$ to sketch the graph of each function. State the $y$-intercept, domain, range, and horizontal asymptote.

(a) $g(x) = 1 + 2^x$

(b) $h(x) = -2^x$

(c) $k(x) = 2^{x-1}$

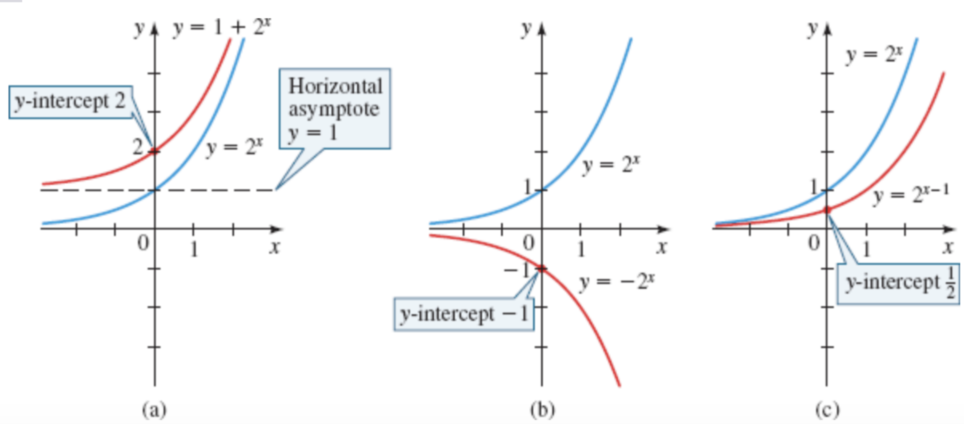
**Solution**

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit to get the graph shown in Figure 3(a). The $y$-intercept is $y = g(0) = 1 + 2^0 = 2$. From the graph we see that the domain of $g$ is the set $\mathbb{R}$ of real numbers, the range is the interval $(1, \infty)$, and the line $y = 1$ is a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect about the $x$-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b). The $y$-intercept is $y = h(0) = -2^0 = -1$. From the graph we see that the domain of $h$ is the set $\mathbb{R}$ of all real numbers, the range is the interval $(-\infty, 0)$, and the line $y = 0$ is a horizontal asymptote.

(c) This time we start with the graph of $f(x) = 2^x$ and shift it 1 unit to the right to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c). The $y$-intercept is $y = k(0) = 2^{0-1} = \frac{1}{2}$. From the graph we see that the domain of $k$ is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

**Figure 3**



After making a couple tables to graph the basic exponential functions $f(x) = 2^x$ and $g(x) = \frac{1}{3}^x$, you'll know all the basic exponential functions $y = a^x$. You should remember their shapes, their domains, ranges, the fact they all pass through $(0, 1)$ and most importantly that all basic exponential graphs $y = a^x$ have the $x-$axis, $y = 0$, as a horizontal asymptote.

After memorizing the basic exponential shapes, it is time to use the method of transformations to get more exponential graphs, like the examples above. When transforming an exponential graph it is most important to determine where the horizontal asymptote is. All exponential graphs have one horizontal asymptote. The exponential function $f(x) = e^x$ with base $e$ is most important. Using base $e$ will make all the formulas in calculus easier. It is best that you get comfortable using it now.

**Exercises**

1. Sketch the graph of $f(x) = 2^{x-3}$. Label its horizontal asymptote on the graph.
2. Sketch the graph of $f(x) = 2^x - 3$. Label its horizontal asymptote on the graph.
3. Sketch the graph of $f(x) = 6 - 3^x$. Label its horizontal asymptote on the graph.
4. Sketch the graph of $f(x) = e^{x-3} + 4$. Label its horizontal asymptote on the graph.

## 4.2 Exponential Functions

All exercises here are similar to the exercises in the previous section.

**Exercises**

1. Sketch the graph of $f(x) = e^x + 3$. Label its horizontal asymptote on the graph.
2. Sketch the graph of $f(x) = e^{-x} - 1$. Label its horizontal asymptote on the graph as a dashed line.
3. Sketch the graph of $f(x) = 10 - e^x$. Label its horizontal asymptote on the graph as a daashed line.

## 4.3 Logarithmic Functions

You should learn to convert a log equation into an exponential equation just as you learned to convert a division problem into a multiplication problem. Some logs simplify like $\log_2 16 = 4$. Most don't like $\log_2 15$. However that's a good thing. $\log_2 15$ is a perfectly good number that we can use to describe the world. Most logs that you'll see in this course will simplify since we want to test your ability to convert to expoenential form and then solve. It is easiest to solve exponential equations when you use a single base.

### Example 1

**Example 1 Logarithmic and Exponential Forms**

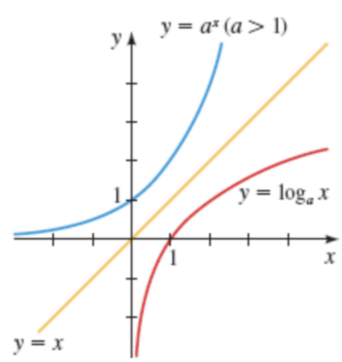
The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

| Logarithmic Form | Exponential Form |
|---|---|
| $\log_{10} 100{,}000 = 5$ | $10^5 = 100{,}000$ |
| $\log_2 8 = 3$ | $2^3 = 8$ |
| $\log_2 \left(\frac{1}{8}\right) = -3$ | $2^{-3} = \frac{1}{8}$ |
| $\log_{1/3} 9 = -2$ | $\left(\frac{1}{3}\right)^{-2} = 9$ |

You should also learn to graph $y = \log_a(x)$ and then transform it. When graphing a log graph, it is most important to find its vertical asymptote of the form $x = a$. All log graphs have one vertical asymptote.

### Example 2

Graph of the logarithmic function $f(x) = \log_a x$




### Exercises

1. Evaluate $\log_4 64$.
2. Evaluate $\log_9 \frac{1}{3}$.
3. Evaluate $\log \frac{1}{1000}$.
4. Evaluate $\log_{16} 1$.

5. Evaluate $\log_{16} \sqrt{2}$.

6. Sketch the graph of $f(x) = \log_2(x - 4)$. Label its vertical asymptote on the graph as a dashed line.

7. Sketch the graph of $f(x) = 1 - \ln x$. Label its vertical asymptote on the graph as a dashed line.

## 4.4 Laws of Logs

You should memorize the laws of logs and learn to use them as you used the laws of exponents in Chapter 1. The laws of logs are derived from the laws of exponents.

### Example 1

Evaluate each expression (without using a calculator).

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 1000$

**Solution**

(a)
$$\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) \qquad \text{Law 1}$$
$$= \log_4 64 = 3 \qquad \text{Because } 64 = 4^3$$

(b)
$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) \qquad \text{Law 2}$$
$$= \log_2 16 = 4 \qquad \text{Because } 16 = 2^4$$

(c)
$$-\frac{1}{3}\log 1000 = \log 1000^{-1/3} \qquad \text{Law 3}$$
$$= \log \frac{1}{10} \qquad \text{Property of negative exponents}$$
$$= -1 \qquad \text{Because } \frac{1}{10} = 10^{-1}$$

**Exercises**

1. Evaluate $\log 4 + \log 25$.
2. Evaluate $\log_2 160 - \log_2 5$.
3. Evaluate $\log_3 \sqrt{27}$.
4. Evaluate $\log_3 100 - \log_3 18 - \log_3 50$.
5. Evaluate $\log_2 8^{33}$.

## 4.5 Exponential and Log Equations

Now that you've learned exponentials and logs, you can use them to solve equations—just as you've used multiplication and division to solve equations. Exponentiation both sides of an equation undoes logs whereas logging both sides of an equations undoes exponentiation—just as dividing both sides of an equation by 3 undoes multiplying by 3. Its that simple, almost. There are some domain issues but its better to ignore them at first. The hard part is that you have to mix these two new techniques (1) exponentiating both sides of an equation and (2) logging both sides of an equation in with all the other techniques you've learned to solve equations since middle school. You must do everything in the right order, i.e. you must follow order of operations. After you solve the equation you should check your solution by plugging it back into the original equation. This will test if you've made any domain mistakes. You'll know your solution is invalid if, when checking, you end up with something like $\log_2(-13)$. Then you have a domain problem and your solution is a false solution that should be disgarded.

### Example 1

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

## Solution 1: Algebraic

Since the base of the exponential term is $e$, we use natural logarithms to solve this equation.

| | |
|---|---|
| $e^{3-2x} = 4$ | Given equation |
| $\ln(e^{3-2x}) = \ln 4$ | Take ln of each side |
| $3 - 2x = \ln 4$ | Property of ln |
| $-2x = -3 + \ln 4$ | Subtract 3 |
| $x = \frac{1}{2}(3 - \ln 4) \approx 0.807$ | Multiply by $-\frac{1}{2}$ |

You should check that this answer satisfies the original equation.

### Example 2

Solve the equation $\log_5(x^2+1) = \log_5(x-2) + \log_5(x+3)$.

## Solution

First we combine the logarithms on the right-hand side, and then we use the one-to-one property of logarithms.

| | |
|---|---|
| $\log_5(x^2+1) = \log_5(x-2) + \log_5(x+3)$ | Given equation |
| $\log_5(x^2+1) = \log_5[(x-2)(x+3)]$ | Law 1: $\log_a AB = \log_a A + \log_a B$ |
| $\log_5(x^2+1) = \log_5(x^2+x-6)$ | Expand |
| $x^2 + 1 = x^2 + x - 6$ | log is one-to-one (or raise 5 to each side) |
| $x = 7$ | Solve for $x$ |

The solution is $x = 7$. (You can check that $x = 7$ satisfies the original equation.)

**Exercises**

1. Solve $4 + e^{5x} = 8$.
2. Solve $\frac{50}{1+e^{-x}} = 4$.
3. Solve $4(1 + 10^{5x}) = 9$.
4. Solve $\log(3x + 5) = 2$.
5. Solve $2 \log x = \log 2 + \log(3x - 4)$.
6. Solve $\log_5(x + 1) - \log_5(x - 1) = 2$.

## 4.6 Modeling with Exponential Functions

Use the following exponential model for all problems in this section.

**Exponential Model**

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where

$$n(t) = \text{population at time } t$$
$$n_0 = \text{initial size of the population}$$
$$r = \text{relative rate of growth (expressed as a proportion of the population)}$$

**Exercises**

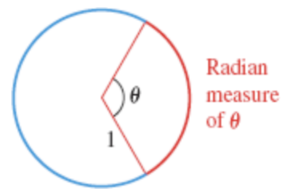
1. A culture of bacteria contains 1500 bacterial and doubles every 30 min. After how many minutes will the culture contain 4000 bacteria. You may leave $e$ or ln in your answer.
2. If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element. You may leave $e$ or ln in your answer.
3. The population of California was $10,586,233$ in 1950 and $23,668,562$ in 1980. Predict the population of California in 2023 assuming its population grows exponentially.

# 6 Trig Functions: Right Triangle Approach

## 6.1 Angle Measure

### Definition of Radian Measure

**Figure 2**



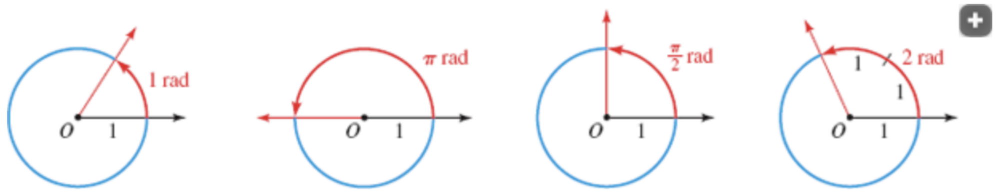

**Definition of Radian Measure**

If a circle of radius 1 is drawn with the vertex of an angle $\theta$ at its center, then the measure of $\theta$ in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle.

## Radian Measure vs. Degree Measure

**Figure 3**

Radian measure




Because a complete revolution measured in degrees is $360^\circ$ and measured in radians is $2\pi$ rad, we get the following relationship between these two methods of angle measurement.

**Relationship Between Degrees and Radians**

$$180^\circ = \pi \, \text{rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \qquad 1^\circ = \frac{\pi}{180} \text{rad}$$

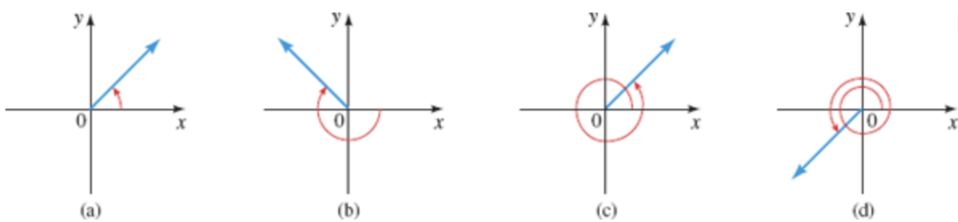
1. To convert degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$.

### Angles in Standard Position

An angle is in **standard position** when it is drawn in the $xy$-plane with its vertex at the origin and its initial side on the positive $x$-axis. Figure 5 gives examples of angles in standard position.

**Figure 5**

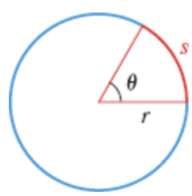
Angles in standard position




Two angles in standard position are **coterminal** if their terminal sides coincide. In Figure 5 the angles in (a) and (c) are coterminal.

### Arc Length Formula

$s = \theta r$




**Length of a Circular Arc**

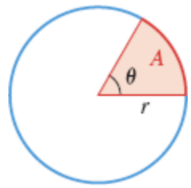
In a circle of radius $r$ the length $s$ of an arc that subtends a central angle of $\theta$ radians is

$$s = r\theta$$

**Sector Area Formula**

**Figure 11**

$$A = \frac{1}{2} r^2 \theta$$




**Area of a Circular Sector**

In a circle of radius $r$ the area $A$ of a sector with a central angle of $\theta$ radians is

$$A = \frac{1}{2} r^2 \theta$$

**Exercises**

1. Find the radian measure of the angle $72°$.

2. Find the degree measure of the angle with radian measure $\frac{11\pi}{3}$.

3. Find the angle between $0°$ and $360°$ coterminal with $733°$.

4. Find an angle between 0 and $2\pi$ that is coterminal with $-\frac{7\pi}{3}$.

5. A central angle $\theta$ in a circle of radius 5 m is subtended by an arc or length 6 m. Find the measure of $\theta$ in degrees.

6. A sector of a circle has central angle $60°$. Find the area of the sector if the radius of the circle is 3 mi.

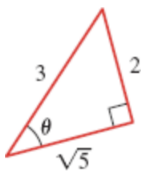
# 6.2 Right Triangle Trig

## Example 1

**Finding Trigonometric Ratios**

Find the six trigonometric ratios of the angle $\theta$ in Figure 3.

**Figure 3**




**Solution**

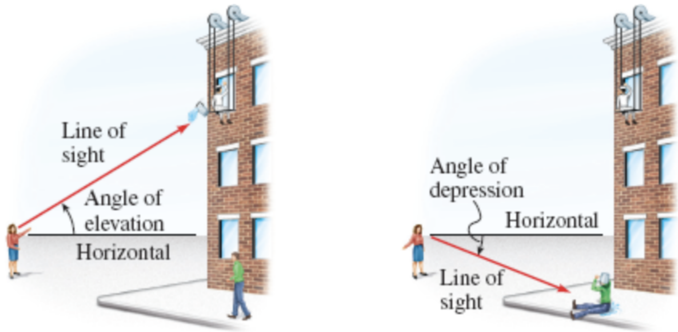
In Figure 3 the side opposite $\theta$ has length 2, the side adjacent $\theta$ has length $\sqrt{5}$, and the hypotenuse has length 3. So, by the definition of trigonometric ratios, we get

$$\sin\theta = \frac{2}{3} \qquad \cos\theta = \frac{\sqrt{5}}{3} \qquad \tan\theta = \frac{2}{\sqrt{5}}$$

$$\csc\theta = \frac{3}{2} \qquad \sec\theta = \frac{3}{\sqrt{5}} \qquad \cot\theta = \frac{\sqrt{5}}{2}$$

### Angle Elevation vs. Angle Depression

**Figure 9**




The next example gives an important application of trigonometry to the problem of measurement: We measure the height of a tall tree without having to climb it! Although the example is simple, the result is fundamental to understanding how the trigonometric ratios are applied to such problems.

## Example 5 Finding the Height of a Tree

A tall tree casts a shadow 532 ft long. Find the height of the tree if the angle of elevation of the sun is $25.7°$.

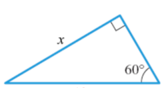
### Solution

Let the height of the tree be $h$. From Figure 10 we see that

$$\begin{aligned} \frac{h}{532} &= \tan 25.7° && \text{Definition of tangent} \\ h &= 532 \tan 25.7° && \text{Multiply by} 532 \\ &\approx 532(0.48127) \approx 256 && \text{Use a calculator} \end{aligned}$$

Therefore the height of the tree is about 256 ft.

### Exercises

1. Find $x$ 


2. From the top of a 220 ft lighthouse the angle of depression to the ocean is 23°. How far

is the ship from the base of the lighthouse?

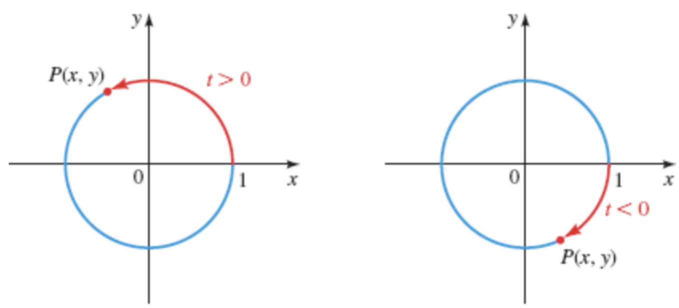
3. A 600 ft guy wire is attached to the top of a communications tower. If the wire makes and angle of 65° with the ground, how tall is the communications tower?

# 5 Trig Functions: Unit Circle Approach

## 5.1 The Unit Circle

### Terminal Points

Figure 2



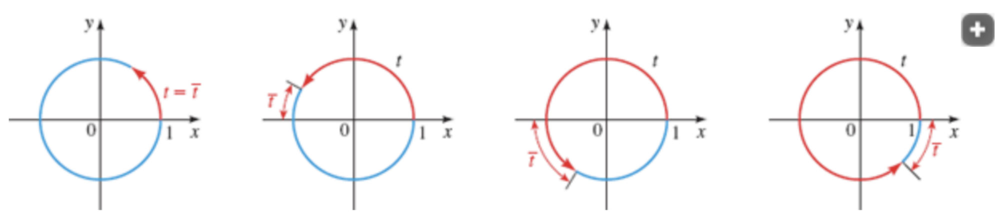

(a) Terminal point $P(x, y)$ determined by $t > 0$

(b) Terminal point $P(x, y)$ determined by $t < 0$

### Reference Angles

Figure 7

The reference number $\bar{t}$ for $t$

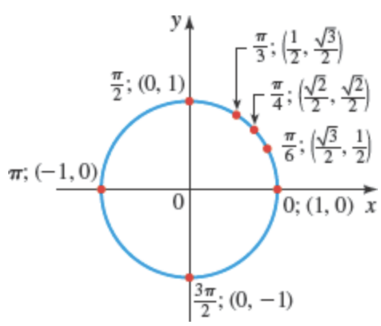


### Reference Table of Special Terminal Points to Memorize

The table gives the terminal points for some special values of $t$.

**able 1**

| $t$ | Terminal point determined by $t$ |
|---|---|
| $0$ | $(1, 0)$ |
| $\frac{\pi}{6}$ | $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ |
| $\frac{\pi}{4}$ | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ |
| $\frac{\pi}{3}$ | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ |
| $\frac{\pi}{2}$ | $(0, 1)$ |
| $\pi$ | $(-1, 0)$ |
| $\frac{3\pi}{2}$ | $(0, -1)$ |



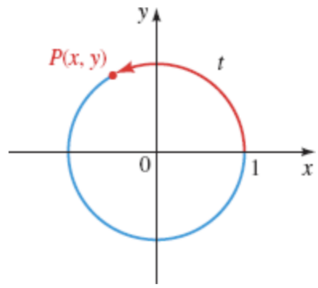

### Exercises

1. Find the point $P(x, y)$ on the unit circle knowing $y = -\frac{1}{3}$ and $P$ is in Quadrant III.
2. Find the reference angle of $t = \frac{5\pi}{6}$.
3. Find the terminal point $P(x, y)$ determined by $t = \frac{-3\pi}{4}$.
4. Find the terminal point $P(x, y)$ determined by $t = \frac{5\pi}{3}$.
5. Find the terminal point $P(x, y)$ determined by $t = \frac{31\pi}{6}$.
6. Find the terminal point $P(x, y)$ determined by $t = \frac{-7\pi}{6}$.

## 5.2 and 6.3 Evaluating Trig Functions

### Definition of Trig Functions in terms of Terminal Points

**Figure 1**




**Definition of the Trigonometric Functions**

Let $t$ be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by $t$. We define

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$

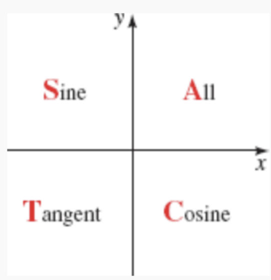
### Memorize the Special Trig Angles

We can remember the special values of the sines and cosines by writing them in the form $\sqrt{\quad}/2$
:

| $t$ | $\sin t$ | $\cos t$ |
|---|---|---|
| 0 | $\sqrt{0}/2$ | $\sqrt{4}/2$ |
| $\pi/6$ | $\sqrt{1}/2$ | $\sqrt{3}/2$ |
| $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ |
| $\pi/3$ | $\sqrt{3}/2$ | $\sqrt{1}/2$ |
| $\pi/2$ | $\sqrt{4}/2$ | $\sqrt{0}/2$ |

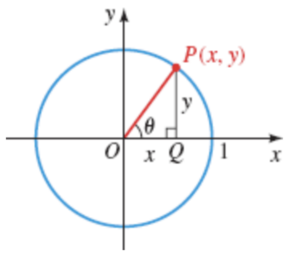
### Memorization Trick to Learn + or -

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.




You can remember this as "All Students Take Calculus."

## Unit Circle Trig and Right Triangle Trig Pictured Together




## Steps to Evaluate Trig Function

**Evaluating Trigonometric Functions for Any Angle**

To find the values of the trigonometric functions for any angle $\theta$, we carry out the following steps.

1. **Find the reference angle.** Find the reference angle $\overline{\theta}$ associated with the angle $\theta$.
2. **Find the Sign.** Determine the sign of the trigonometric function of $\theta$ by noting the quadrant in which $\theta$ lies.
3. **Find the Value.** The value of the trigonometric function of $\theta$ is the same, except possibly for sign, as the value of the trigonometric function of $\overline{\theta}$.

## Exercises

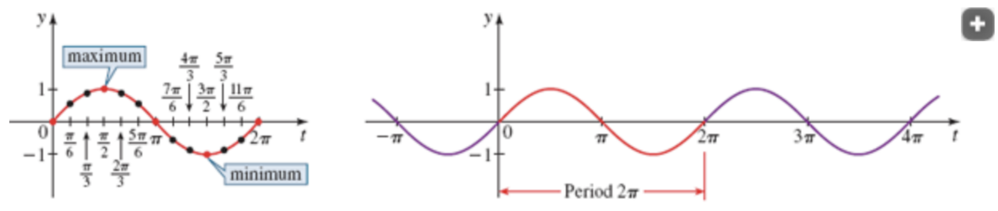
1. Find $\sin 225°$
2. Find $\cos 135°$
3. Find $\tan(-60°)$
4. Find $\cos 660°$
5. Find $\sin \frac{5\pi}{3}$
6. Find $\cos(-\frac{7\pi}{3})$
7. Find $\tan \frac{5\pi}{6}$
8. Find the values of all six trig functions of $\theta$ when $\sin\theta = \frac{3}{5}$ and $\theta$ in quadrant II.
9. Find the values of all six trig functions of $\theta$ when $\tan\theta = -\frac{7}{12}$ and $\cos\theta > 0$.

## 5.3 Trig Graphs (Sin and Cos Only)

### Sine Graph

**Figure 2**

Graph of $\sin t$

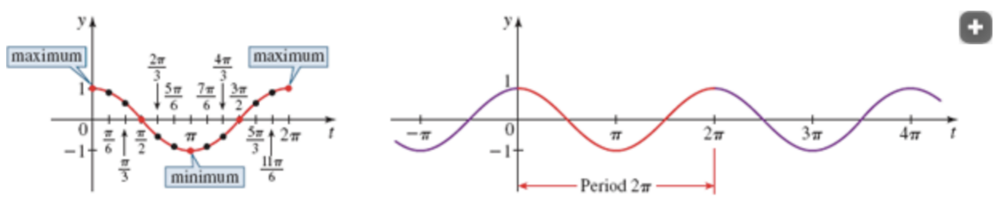



(a) One period of $y = \sin t$, $0 \le t \le 2\pi$ (b) Graph of $y = \sin t$

### Cosine Graph

**Figure 3**

Graph of $\cos t$




(a) One period of $y = \cos t$, $0 \le t \le 2\pi$ (b) Graph of $y = \cos t$

### Exercises

1. Sketch the graph $f(t) = -4\cos 2t$.
2. Sketch the graph $f(t) = \frac{1}{2}\sin\frac{1}{2}t$.
3. Sketch the graph $y = 2\sin(x - \frac{\pi}{3})$.
4. Sketch the graph $y = -\cos(x + \frac{\pi}{2})$.
5. Sketch the graph $y = -4\sin(2x + \pi)$.

## 5.4 Inverse Trig (Sin, Cos, and Tan Only)

### Definition Sine Inverse or Arcsin

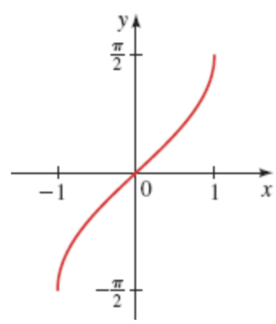
The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

### Graph of Sine Inverse

Graph of $y = \sin^{-1} x$




### Definition Cosine Inverse or Arccosine

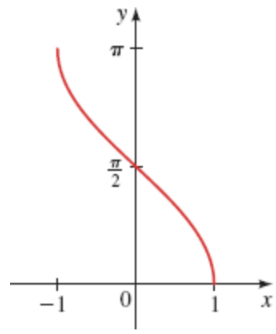
The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

### Graph of Cosine Inverse

Graph of $y = \cos^{-1} x$




### Definition Tangent Inverse or Arctan

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

### Exercises

1. Find $\cos^{-1}(\frac{1}{2})$.
2. Find $\sin^{-1}(-\frac{\sqrt{2}}{2})$.
3. Find $\tan^{-1}(-\sqrt{3})$.
4. Find $\tan\left(\cos^{-1}(\frac{-\sqrt{2}}{2})\right)$.
5. Find $\cos\left(\sin^{-1}(\frac{\sqrt{3}}{2})\right)$.

# 7 Analytic Trig

## 7.1 Trig Identities

There is not one way to verify a trig identity. There are many. This freedom irritates many students. Many students prefer to have sequential rules to follow when solving a math problem. Since there are no set rules to follow when verifying trig identities, you'll have to practice a lot of verifications until you get the hang of it. There is one fundamental identiy one should memorize $\sin^2 t + \cos^2 t = 1$. After that it is a matter of taste how many of the "fundamental" trig identities you decide to memorize. There are no set rules to trig verifications but there are some guildlines.

**Trig Identity Verification Guildlines**

### Guidelines for Proving Trigonometric Identities

1. **Start with One Side.** Choose one side of the equation, and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use Known Identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to Sines and Cosines.** If you are stuck, you may find it helpful to first rewrite all functions in terms of sines and cosines.

## Example 1

**Proving an Identity by Combining Fractions**

Verify the identity

$$2\tan x\sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$$

**Solution**

Finding a common denominator and combining the fractions on the right-hand side of this equation, we get

$$\begin{aligned}
\text{RHS} &= \frac{1}{1-\sin x} - \frac{1}{1+\sin x} \\
&= \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)} && \text{Common denominator} \\
&= \frac{2\sin x}{1-\sin^2 x} && \text{Simplify} \\
&= \frac{2\sin x}{\cos^2 x} && \text{Pythagorean identity} \\
&= 2\frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) && \text{Factor} \\
&= 2\tan x\sec x = \text{LHS} && \text{Reciprocal identities}
\end{aligned}$$

## Example 2

Verify the identity $\dfrac{1+\cos\theta}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$.

### Solution

We prove the identity by changing each side separately into the same expression. (You should supply the reasons for each step.)

$$\text{LHS} = \frac{1+\cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta} = \sec\theta + 1$$

$$\text{RHS} = \frac{\tan^2\theta}{\sec\theta-1} = \frac{\sec^2\theta - 1}{\sec\theta - 1} = \frac{(\sec\theta-1)(\sec\theta+1)}{\sec\theta-1} = \sec\theta + 1$$

It follows that LHS = RHS, so the equation is an identity.

**Exercises**

1. Verify $\frac{\tan\theta}{\sec\theta} = \sin\theta$.

2. Verify $\frac{\cos\theta}{\sec\theta\sin\theta} = \csc\theta - \sin\theta$.

3. Verify $(\sin t + \cos t)^2 = 1 + 2\sin t\cos t$.

4. Verify $\frac{1}{1-\sin^2 x} = 1 + \tan^2 x$.

5. Verify $\frac{1-\sin t}{1+\sin t} = (\sec t - \tan t)^2$.

## 7.2 Addition and Subtraction Formulas

In addition to the truly fundamental trig identity $\sin^2 t + \cos^2 t = 1$, you should memorize the following addition formulas:

1. $\sin(s+t) = \sin s\cos t + \sin t\cos s$

2. $\cos(s+t) = \cos s\cos t - \sin s\sin t$.

as well as the even or odd identities that easily seen from the graphs of sin and cos

1. $\sin(-s) = -\sin s$

2. $\cos(-s) = \cos s$.

If you also know $\tan t = \frac{\sin t}{\cos t}$, that is all you need. All the remaining trig identities reduce to these, including the double and half angles formulas in the next sections. A book like ours encourages students to memorize a bunch of unnecessary trig identities, ones that are easily derived from the basic ones above. If you have a weak memory like I do, it is better to just memorize a few identities and start doing many examples.

## Example 1

**Using the Addition and Subtraction Formulas**

Find the exact value of each expression.

(a) $\cos 75^\circ$

(b) $\cos \frac{\pi}{12}$

**Solution**

(a) Notice that $75^\circ = 45^\circ + 30^\circ$. Since we know the exact values of sine and cosine at $45^\circ$ and $30^\circ$, we use the Addition Formula for Cosine to get

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

(b) Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\begin{aligned}\cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

## Example 2

Evaluate $\sin(\theta + \phi)$, where $\sin\theta = \frac{12}{13}$ with $\theta$ in Quadrant II and $\tan\phi = \frac{3}{4}$ with $\phi$ in Quadrant III.

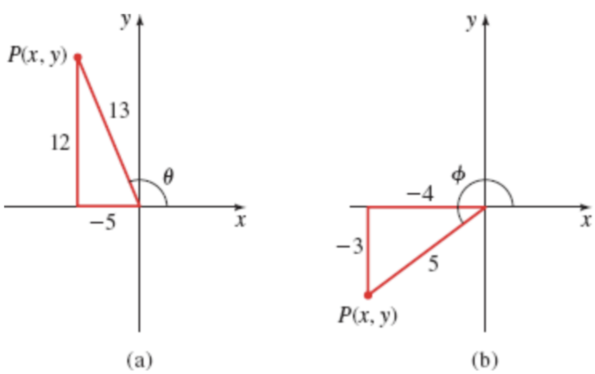
**Solution**

We first sketch the angles $\theta$ and $\phi$ in standard position with terminal sides in the appropriate quadrants, as shown in Figure 4. Since $\sin\theta = y/r = \frac{12}{13}$, we can label a side and the hypotenuse in the triangle in Figure 4(a). To find the remaining side, we use the Pythagorean Theorem.

$$\begin{aligned} x^2 + y^2 &= r^2 && \text{Pythagorean Theorem} \\ x^2 + 12^2 &= 13^2 && y = 12,\ r = 13 \\ x^2 &= 25 && \text{Solve for } x^2 \\ x &= -5 && \text{Because } x < 0 \text{ in Quadrant II} \end{aligned}$$

Similarly, since $\tan\phi = y/x = \frac{3}{4}$, we can label two sides of the triangle in Figure 4(b) and then use the Pythagorean Theorem to find the hypotenuse.

**Figure 4**




Now, to find $\sin(\theta + \phi)$, we use the Addition Formula for Sine and the triangles in Figure 4.

$$\begin{aligned} \sin(\theta + \phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi && \text{Addition Formula} \\ &= \left(\frac{12}{13}\right)\left(-\frac{4}{5}\right) + \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) && \text{From triangles} \\ &= -\frac{33}{65} && \text{Calculate} \end{aligned}$$

## Exercises

1. Evaluate $\sin 75°$.
2. Evaluate $\cos 195°$.

3. Evaluate $\sin \frac{17\pi}{12}$.
4. Evaluate $\cos \frac{-5\pi}{12}$.
5. Evaluate $\tan \frac{-\pi}{12}$.
6. Verify $\sin(x - \frac{\pi}{2}) = -\cos x$.
7. Verify $\sin(x - \pi) = -\sin x$.

## 7.3 Double Angle and 1/2 Angle Formulas

You only need to memorize the double and $\frac{1}{2}$ angle formulas for sin and cos. However, as was said in the previous section, these formulas can be easily derived from the angle addition formulas, which really should be memorized. You can decide for yourself whether you should memorize or derive the double and $\frac{1}{2}$ angle formulas for sin and cos. You will never be tested on the tan double and $\frac{1}{2}$ angle formula.

### Example 1

**Using a Half-Angle Formula**

Find the exact value of $\sin 22.5°$.

**Solution**

Since $22.5°$ is half of $45°$, we use the Half-Angle Formula for Sine with $u = 45°$. We choose the + sign because $22.5°$ is in the first quadrant.

$$\sin \frac{45°}{2} = \sqrt{\frac{1 - \cos 45°}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1 - \sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2 - \sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \frac{1}{2}\sqrt{2 - \sqrt{2}} \qquad \text{Simply}$$

### Exercises

1. Use $\sin x = \frac{5}{13}$ to find $\sin 2x$, $\cos 2x$, if you also know $x$ is in quadrant I.
2. Use $\sin x = \frac{-4}{5}$ to find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, if you also know $x$ is in quadrant II.
3. Find $\sin 15°$
4. Find $\cos 112.5°$

5. Evaluate $\cos\frac{\pi}{8}$.

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$$\sin\frac{45°}{2} = \sqrt{\frac{1-\cos 45°}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}} \qquad \text{Simply}$$

### Exercises

1. Use $\sin x = \frac{5}{13}$ to find $\sin 2x$, $\cos 2x$, if you also know $x$ is in quadrant I.
2. Use $\sin x = \frac{-4}{5}$ to find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, if you also know $x$ is in quadrant II.
3. Find $\sin 15°$
4. Find $\cos 112.5°$
5. Evaluate $\cos\frac{\pi}{8}$.

## 7.4 Basic Trig Equations

### Trig Identities vs. Trig Equations

An equation that contains trigonometric functions is called a **trigonometric equation**. For example, the following are trigonometric equations:

$$\sin^2\theta + \cos^2\theta = 1 \qquad 2\sin\theta - 1 = 0$$

The first equation is an *identity*—that is, it is true for every value of the variable $\theta$. The second equation is true only for certain values of $\theta$. To solve a trigonometric equation, we find all the values of the variable that make the equation true.

## Example 1

Find all solutions of each equation.

(a) $2\sin\theta - 1 = 0$

(b) $\tan^2\theta - 3 = 0$

## Solution

(a) We start by isolating $\sin\theta$.

$$\begin{aligned} 2\sin\theta - 1 &= 0 && \text{Given equation} \\ 2\sin\theta &= 1 && \text{Add 1} \\ \sin\theta &= \frac{1}{2} && \text{Divide by 2} \end{aligned}$$

This last equation is the equation we solved in Example 1. The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where $k$ is any integer.

(b) We start by isolating $\tan\theta$.

$$\begin{aligned} \tan^2\theta - 3 &= 0 && \text{Given equation} \\ \tan^2\theta &= 3 && \text{Add 3} \\ \tan\theta &= \pm\sqrt{3} && \text{Take the square root} \end{aligned}$$

Because tangent has period $\pi$, we first find the solutions in any interval of length $\pi$. In the interval $(-\pi/2, \pi/2)$ the solutions are $\theta = \pi/3$ and $\theta = -\pi/3$. To get all solutions, we add integer multiples of $\pi$ to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad \theta = -\frac{\pi}{3} + k\pi$$

## Exercises

1. Find all solutions $x$ of $\sqrt{2}\cos x - 1 = 0$ for $0° \leq x \leq 360°$.
2. Find all solutions $x$ of $\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$.
3. Find all solutions $t$ of $\sqrt{3}\tan t + 1 = 0$ for $0° \leq t \leq 360°$.

4. Find all solutions $\theta$ of $2\cos^2\theta - 1 = 0$ for $0 \leq \theta \leq 2\pi$.