In Defense of "Mindless Rote"

Ethan Akin
Mathematics Department
The City College
137 Street and Convent Avenue
New York City, NY 10031
ethanakin@earthlink.net

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> It is a profoundly erroneous truism repeated by all copybooks, and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of operations which we can perform without thinking about them. Operations of thought are like cavalry charges in battle - they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.
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> -Alfred North Whitehead, *Introduction to Mathematics*

In *The Schools We Need and Why We Don't Have Them* E. D. Hirsch provides the quote which will be our Scripture passage for today's sermon. Following Whitehead, I propose to defend not thinking, to consider the relationship between thinking and not thinking and to describe how symbolism - particularly in mathematics - facilitates not thinking. Above all, I want to argue that all this avoidance of thought is a Good Thing.

In some areas all this is noncontroversial, even obvious. Consider such practices as cooking, carpentry, playing a musical instrument, horseback riding and other sports. Each builds upon a foundation of physical skills and in each case mastery consists of performing with automatic facility. As a beginner you move slowly, thoughtfully, with conscious attention. In a disciplined way you repeat the same movements again and again. Think of Audrey Hepburn at the cooking school in *Sabrina*: "one-two-three, crack. New egg. One-two-three, crack. New egg..." Think of the scales and arpeggios with which as a budding pianist you train your hands. As you practice, you speed up and your movements alter so that they are less in your mind than "in your fingers". The skill is gradually incorporated into muscle memory. Similarly when you learn a new piece, you move via repeated practice from conscious attention to unconscious mastery. The transformation is not irrevocable. When you discover yourself repeatedly making an error in a passage or you need to change the fingering, you slow up again, consciously overriding the automatic response and practicing until the correct procedure has replaced the old one. That is, the movement, lifted up for conscious repair, is now allowed to sink back down into your fingers.

Understanding just what you are doing and why you are doing it is not essential in learning these skills. It can even be an impediment if it is regarded as a substitute for the boring repetition that practicing a skill requires. Someone who "knows how to hold a pool cue" probably doesn't, if he hasn't practiced much shooting. As Yogi Berra is supposed to have said: "In theory, theory and practice are the same thing. But in practice they aren't."

The understanding provided by an explanation can be helpful, but primarily as motivation for the discipline required. When the teacher says: "Use this fingering instead of that one." Your question: "Why should I change?" is legitimate, especially as you are probably used to the old one and so find the effort of changing to be a bother. The teacher explains: "For a simple passage it doesn't make much difference, but for the kinds of complicated passages that you will get to later, you can see that what you are doing will be awkward and will break the flow. The point is to develop the right habit now so that you don't have to change it later." This sort of helpful justification is good teaching, much better than: "Because I say so." But in any case the student does the correction work attending to the action rather than to its purpose.

Of course, all of these arts involve thinking, but the thought occurs at higher levels which are built upon a foundation of unthinking facility. You think about how to vary a sauce not how to crack an egg, about what is the appropriate emphasis for a musical passage not what note is flat in the key of F. In competitive sports your tactical decisions assume the background skills are in place.

Horseback riding provides an especially interesting example of the relationship between thought and automatic muscle skills. Equitation underwent a full-scale Kuhnian revolution in the early twentieth century when an Italian cavalry captain named Caprilli began thinking seriously about the dynamics of the horse and horse-plus-rider in motion. The result was the modern Forward Seat or Hunt Seat. Manuals like Vladimir Littauer's *Common Sense Horsemanship* spend a great many pages explaining why the rider should hold his body in particular ways. Since proper riding posture is hard to develop, all this explanation provides motivation in the form of helpful justification. But you still have to practice and practice, ride and ride, so that your body will ride the horse. There is a great deal of New Age, "Use the Force, Luke" talk about feeling and instinct in horsemanship, especially because you are trying to learn to feel the horse's body as well as your own. But again, correction requires conscious attention. "How many times have I told you? Don't lean your body like that when you take off into a canter. Your weight unbalances the horse." "O yes. Got to remember. Think. Think."

All these have been examples of physical skills. Even granting all I have said, you might argue that none of this applies to the mental activities which are the concerns of English and Mathematics teachers. I claim, instead, that learning to read and use symbolic systems are mental analogues of the physical skills considered above. Success at learning the alphabet, for example, consists in recognizing the letters instantly without conscious effort. A dyslexic can pause and work out the difference between a "d" and a "b". What is lacking is the automatic recognition response which easy facility in reading requires. This mirror symmetry between different letters illustrates that avoidable flaws can occur in symbolic systems and then be retained by tradition. Such symmetries do not occur in the Cyrillic alphabet. I wonder if it affects the frequency or severity of dyslexia among readers of Russian.

The issue of rote facility, or lack thereof, lurks in the background even in relatively advanced courses. Teachers of elementary calculus will recognize certain excuses that students give after bad test results: "I really understand it, but ..." and "I just didn't have enough time." In this respect doing calculus is like shooting pool. It is not enough to see the logic of the Product Rule or Chain Rule. It is necessary to do a lot of drill problems as homework so that the use of the routines becomes automatic for uncomplicated examples. This is also the reason that tests are given with time limits. Speed per se is not that important. However, just as with piano playing, performance with a reasonable amount of speed requires that the steps begin to move out of conscious control. It is this automatic facility which is being tested for. A slow, stumbling performance often indicates a lack of practice.

The Order of Operations Rules provide a good example of the need for unconscious adaptation comparable to a feel for the horse. The expression $2 \ + \ 3 \ \cdot \ 5$ is ambiguous, an ambiguity which can be resolved by using parentheses. Thus,

$$(2 \ + \ 3) \ \cdot \ 5 \quad = \quad 5 \ \cdot \ 5 \quad = \quad 25$$

while

$$2 \ + \ (3 \ \cdot \ 5) \quad = \quad 2 \ + \ 15 \quad = \quad 17.$$

To reduce the number of parentheses, which can become cumbersome in complicated expressions, a new rule is introduced into the symbolism. When written without parentheses the expression is defined to mean either the first expression yielding 25 or the second yielding 17. The new rule is a convention which is then built into the symbolism. If the rule were "always do the operations in order from left to right" then the ambiguity would resolve to the 25. However, what has been chosen is the rule "multiplication has priority over addition" and so without parentheses the expression means multiply first to get 17. Students are taught this rule but when it is learned properly it is rarely consciously applied. Instead, you are trained to read and write the symbols so that multiplication provides a tighter linkage between the symbols than addition does. When your eye has been trained properly you see the expression as though it had been written $2 \quad + \quad 3 \cdot 5$ and to the trained eye an expression written $2 + 3 \quad \cdot \quad 5$ "looks wrong". When extending the rules to include exponents, for example, you tend to refer back to the formal list of Order of Operations but after some practice these new extensions are also incorporated into your use of the symbolism.

This example illustrates that all sorts of complications and subtleties are hidden within effective symbol systems. That is why they require a lot of practice but it is also why their use is so effective. Like a well charged battery such a system has the capacity to store a lot of energy which can then provide considerable light when used properly. Just as the student rider does not need to know about the subtle concerns which lead to the development of the Forward Seat, so also the beginning math student need only learn to recognize and to properly manipulate the symbols. In each case, the teacher should know more about the hidden issues and at times it is helpful for the student to take note of them as well.

There is an obvious answer to all this. You don't have to worry about how to sit a horse if you plan to travel exclusively by car or plane. Why should a student who is equipped with a calculator be subjected to the tedium of learning the multiplication tables and the associated algorithm? Being able to solve Mixture Problems is not especially important for most people, but the understanding of units of measurement, obtained by thinking about such problems, can be quite useful. Granted that the symbol manipulation allows the student to avoid thinking, doesn't this mean that using such systems circumvents the very lessons we should be teaching? Like the arcana of the Forward Seat the specialized language of algebra should perhaps be reserved for the perverse few who take an interest in that sort of thing.

All this means that instead of merely brandishing Whitehead's quote I will have to defend it in detail. To make my case I will consider some illustrative bits of mathematical symbolism.

When I was in elementary school we were told that long division was considered an advanced subject back in the Middle Ages. This had the desired effect of motivating us to learn the long division algorithm and instilling in us great pride when we succeeded. Our sense of superiority to all those monks was of course misplaced. Instead we should have been feeling appreciation and gratitude for the Arabic number system.

The most direct solution to the problem of written notation for numbers is to put down the correct number of marks, e.g. represent 17 by seventeen vertical strokes. This is inefficient for numbers beyond ten or so but even for smaller numbers you run into what I will call the "pigeon problem". Pigeons, like human beings, can distinguish at a glance between a group of two and a group of three. They cannot distinguish between a group of seven and a group of nine. Humans can but they have to count (unless the groups are arranged in recognized conventional patterns like the faces of playing cards). A common solution for low level counting is to group the vertical strokes and to write every fifth as an angled slash over groups of four. The pigeon problem reasserts itself once you get past thirty-five or so because then you have more than seven blocks of five. Roman numerals solve the problem by using conventional symbols for large size groups plus a few simple rules. With a little practice (required of elementary school students in an era before mine) you can recognize at a glance the number so represented. However, you will need new symbols as the numbers get bigger and, more seriously, the notation is not well adapted to arithmetic operations. All these problems are solved by the Arabic number system with zero as a place holder. (The pigeon problem recurs for the Arabic number system to be solved by scientific notation and it could in principle recur even for scientific notation but the numbers where the problems arise are larger than science is required to deal with, e. g. $10 \wedge (10 \wedge (10 \wedge (10 \wedge 10)))$.) Notice that the problem that number notation tries to solve is: how can I recognize a number in the same immediate way that I read a word?

It is worthwhile showing students the physical basis for arithmetic operations. Addition comes from the observation that, in general, the count obtained when you push together two heaps of objects depends only on the count in each heap and not on the contents (except when you mash some things together or something falls off the table). Multiplication results from repeated addition: "As I was going to St. Ives, I met a man with 7 wives. Every wife had 5 sacks. Every sack had 3 cats and every cat had 10 kits. How many kittens are there?" Similarly, subtraction is introduced as "takeaway" (the reverse of pushing together for addition) and division is introduced with "gazinta". (As in 3 gazinta 15 but when 3 gazinta 17 there are 2 left over. Notice that after a while you slide into saying there *is* 2 left over, referring to the number which is the remainder rather than to the objects being counted.)

Having introduced such operations physically, it is important to introduce the symbolic notation as well and to insist that the students memorize the single digit addition rules and times tables. In aid of this, flash cards are a dandy device (O horrors). For some reason the prejudice has arisen that to demand students memorize poems or patterns is a form of child abuse. In fact, the ability to commit things to memory is a useful skill and like every such skill it improves with practice.

As students learn the notation of arithmetic they unconsciously learn various properties which are hidden in the notation. It is useful to let such hidden aids do their work. For example, the commutative law of addition: $5 + 2 \quad = \quad 2 + 5$ is obvious from the physically symmetric way that addition is defined. On the other hand, the commutative law of multiplication $5 \cdot 2 \quad = \quad 2 \cdot 5$ is not obvious. However, students will think it is because of the similarity between addition and multiplication notation. Let them. There is no particular reason to use the rectangle picture of multiplication in the beginning. Its purpose, grouping first by rows and then by columns, is to provide a proof of the commutative law for multiplication. However, one should never go through the proof of a result which is (1) true and (2) regarded as obvious by the class. The time to raise the issue of commutativity is when exponents are introduced. One can then note that the commutative law for exponentiation, i.e. repeated multiplication, is false while for multiplication, i.e. repeated addition, it is true. This comparison has no punch unless the students have been using for years the fact that the order doesn't matter in multiplication. In fact, you can get away without mentioning it even when exponents come up because the superscript notation for exponentiation is so asymmetric that students don't expect commutativity (at least until the notation $2 \wedge 3$ is introduced).

An example of an especially nasty dog that you want to leave sleeping beneath a blanket of notation is the different meanings of the minus sign. I didn't notice, until I had been teaching for years and acquired my first calculator, that the symbol " - " has three different meanings: (1) subtraction, a binary operation, e.g. $5 - 2$, (2) the negative sign which together with the absolute value comprises a negative signed number, e.g. -2, (3) changing the sign, a unary operation, as in $-x$. Once you notice this, you realize that the rule for subtracting signed numbers is just like the rule for dividing fractions. That is, to divide two fractions, you convert to multiplication by inverting the second fraction, i.e. multiply by the reciprocal. Similarly, here is the pedantic procedure for $3 - (-2)$: In order to subtract negative 2, convert to addition by changing the sign of negative 2 to obtain positive 2 which, when added to 3 yields 5. This is an unnecessary mental exertion and a waste of some useful ambiguity. The reason that the same symbol is used for the three different meanings is so you can blur them. The way to think instead of $-(-2)$ is as minus signs multiplied onto the following number. Then minus times a minus is plus . . .

Fractions are introduced by changing the physical model of division from the gazinta picture, separation of a heap into a number of smaller heaps of equal count, to the division of a continuous quantity like a line or a circle, e. g. a pie, into pieces of equal size. The fraction notation deliberately blurs the two pictures. The residue of the two models is revealed in the two different ways of reading the fraction $\frac{2}{3}$ i.e. as "two over three" or as "two thirds". As before, I believe that in teaching arithmetic of fractions you should let the symbolic notation carry the student along. Thus, multiplication of fractions should be regarded as easy because you do just what the notation suggests you should do:

$$\frac{2}{3} \times \frac{5}{7} \qquad = \qquad \frac{10}{21}.$$

But the logical rule for addition for addition of fractions ought to be:

$$\frac{2}{3} + \frac{5}{7} \qquad = \qquad \frac{7}{10}.$$

So you use the pie model to explain why the addition rule doesn't work the nice way it "should". That is, $2\ fifths\ +\ 1\ fifth$ is $3\ fifths$ because you are counting $fifths$, i.e. the top of the fraction is the count and the bottom is the units you are counting, just like $2\ inches\ +\ 1\ inch$ is $3\ inches$. Then the complicated LCD rule is explained as converting to common units, as in $2\, inches\ +\ 1\, foot$ equals $2\, inches\ +\ 12\, inches$ and so is $14\, inches$. Once again, this sort of explanation provides the motivation for learning the relatively complicated routine involved in adding fractions. But just as the piano teacher's explanation for the change in fingering does not substitute for the practice of learning the new pattern, so too, the routine of addition of fractions has to be acquired by practicing a lot of examples until it becomes as automatic as a passage of music. Similarly, the "invert and multiply" rule can and should be explained but then has to become an automatic routine.

The same sort of natural relationship between explanation and drill occurs in algebra. The classical picture of an equation as a balance scale justifies the rules used for solving equations but facility in solving them requires practice.

I believe that most mathematicians share my belief that systems like the Arabic number notation with the associated algorithms for multiplication and division, and the symbolisms of fractions and of algebra are really triumphs of human ingenuity and that to learn them is to acquire tools of great beauty as well as power. We strongly feel that their use should be encouraged rather than avoided. However, this kind of preaching appeals only to the already converted. On the other hand, there are much more utilitarian reasons for acquiring these techniques.

My real defense of all this symbolic manipulation is that it is easy. I hasten to add that when I speak of solving a system of two simultaneous linear equations in two unknowns as easy, I am using the word "easy" as a term of art. None of this stuff is easy when you start learning it. But these routines all have the capacity to become easy given disciplined practice. They are easy after they have become automatic. Furthermore, this is the way students can and do react after they have learned it. Looking back they should be thinking: "That stuff from six months ago is really easy. I can't remember why I thought it was so hard. Now this new stuff though..."

What is hard is thinking. Despite the initial quote, neither Whitehead nor I really intend to disparage thinking. The algorithms and algebra routines are resources which can be deployed to help with problems where thought is required.

For example, I think it is safe to say that the hardest part of elementary algebra is the so-called word or story problems. What is hard about them is the thinking required to interpret the verbal descriptions. The student succeeds by a process of translation into algebra by which the problem is reduced to an algebraic equation. The term "reduce" here means replace the original verbal statement by an algebra problem. This will only work when the student regards the algebra as easy, at least by comparison.

There are many examples where thinking about complicated matters builds upon a foundation of easy familiarity with earlier algebra. For example, learning to manipulate units is fundamental in scientific research but also in the practices of nursing and cooking. The key is to recognize that the word "per" (Latin for "through") always means "divide". For example, "miles per hour" means "miles divided by hours" or just "miles over hours". Then conversion to "feet per second" uses the fact that units cancel just the way numbers do in multiplication of fractions. This makes conversions of units fairly straightforward to deal with but only for students who are comfortable with the routines of multiplying fractions.

Next, consider 2 . The calculator says it is 1.4142135623731 and when you check by squaring the calculator gives you 2 . But the classic long multiplication rule reveals that this cannot be exactly correct.

$$
\begin{array}{r r}
 & 1.4142135623731 \\
\times & 1.4142135623731 \\
\hline
 & 14142135623731 \\
 & \ldots\ \ 193\ \\
 & \ldots.\ \quad \\
 & \ldots.\ \quad \\
\hline
 & \ldots.\ \ 1
\end{array}
$$

So when you actually multiply out what you get is a long decimal ending with a 1 . Since it is neither 2.0000 nor 1.99999... the big thing you squared is not exactly the square root of 2. This argument requires that the student remembers the long multiplication routine.

Not everything that you learn has to be at your fingertips. However, there are lots of tricks that you have to learn well the first time so that when you need them later you can easily relearn them. The method of Completing the Square is learned and then gratefully forgotten after it has been used to get the Quadratic Formula. However, the return of the repressed occurs first in analytic geometry and later in some integration techniques. You have to be able to say "O yeah. How did that go again?" and then dust it off with a few examples once it is retrieved from storage in the mental attic.

On the other hand, mathematics is cumulative and there are a great many skills that you have be unthinkingly familiar with. Every grumpy calculus teacher will tell you that most of the problems his students have come from weaknesses in algebra. For the students who say "I really understand it but...." the but is that for them algebra is not easy background knowledge. They are trying to build on a foundation of dust. A lot of college majors need a bit of calculus or statistics which are simply walled off to students who don't have sufficient skills in algebra. These are basically not hard subjects but they appear unnecessarily terrifying to such students.

Conversely, a practiced facility with algebra can provide its own positive reinforcement. Not only is the mathematics built on the algebra, but facility in algebra gives the student confidence in the face of new mathematical challenges. As the above discussion makes clear such confidence is entirely justified.

After this salvo of opinions perhaps some diplomacy would not be out of place. These education debates arise from serious disagreements about matters which we all regard as important. However, with an optimistic attitude nourished by complete inexperience with actual political struggles, I believe that some grounds for compromise exist with what my high school debate teammates would call "my worthy opponents".

Think of the pictures and stories that move them both positively and negatively. The bad first: the kind of soul-destroying teaching portrayed in Dickens' Hard Times with students pounding away at mind-numbing, hateful, tedious tasks, working in a spirit not so different from the exploited children of nineteenth century factories. (I can't resist quoting a little poem by Sarah Cleghorn:

The golf links lie so near the mill
That almost every day
The laboring children can look out
And see the men at play.)

Who doesn't want to get away from that?

The positive vision has a common appeal as well. The hope is to try to tap into the natural desire that children have to learn. The title of Gopnik, Meltzoff and Kuhl's delightful study The Scientist in the Crib is deliberately ambiguous as they document how even preverbal infants investigate the world around them. How much might be accomplished could we but stimulate in class the mongoose-like curiosity that children naturally have. (Think of the motto of Kipling's Rikki-Tikki-Tavi: "Run and find out.")

We traditionalists may insist on the value of drill but we don't have a commitment to making education boring and hateful. We don't accept the judgment of my old elementary school pal who said to me, when we saw the section in our English book labeled Spelling Can Be Fun, "Spelling isn't supposed to be fun." Instead, we hope to organize the work in ways that make it interesting. Imagination is needed to design intellectually serious education which is also exciting, but examples do exist. As for the drill and practice, some of which is simply essential, we hope to convince the students as well as the teachers that all this work will pay off. One of my old German teachers, a sweet man, inappropriately named Anger, used to tell us "You will love it when you have learned the prepositions that take the Dative." Wise guys all, we used to pretend we were uncertain whether he was giving us a prediction or an order. The analogue for algebra of his message is what we are trying to get across. But in addition, we have to maintain for Mathematics, as firmly as he did for German, that success requires Sitzfleisch.

I believe that we can successfully make the case that algorithms and symbolic manipulation provide good value. That is, the benefits of these skills are quite substantial. The final argument will then occur over the cost in time and sweat which it is reasonable to exact from students in order for them to acquire these goods. An agreeable balance will be hard to achieve, but I hope that we can simplify the arguments so that in the end we are (in the words of the old joke the details of which I will omit) "just haggling over price."