**T2.** ***Sylvester's inequality*** states that if $A$ and $B$ are $n \times n$ matrices with rank $r_A$ and $r_B$, respectively, then the rank $r_{AB}$ of $AB$ satisfies the inequality

$$r_A + r_B - n \leq r_{AB} \leq \min(r_A, r_B)$$

where $\min(r_A, r_B)$ denotes the smaller of $r_A$ and $r_B$ or their common value if the two ranks are the same. Use your technology utility to confirm this result for some matrices of your choice.

# 4.9 Basic Matrix Transformations in $R^2$ and $R^3$

In this section we will continue our study of linear transformations by considering some basic types of matrix transformations in $R^2$ and $R^3$ that have simple geometric interpretations. The transformations we will study here are important in such fields as computer graphics, engineering, and physics.

There are many ways to transform the vector spaces $R^2$ and $R^3$, some of the most important of which can be accomplished by matrix transformations using the methods introduced in Section 1.8. For example, rotations about the origin, reflections about lines and planes through the origin, and projections onto lines and planes through the origin can all be accomplished using a linear operator $T_A$ in which $A$ is an appropriate $2 \times 2$ or $3 \times 3$ matrix.

## Reflection Operators

Some of the most basic matrix operators on $R^2$ and $R^3$ are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called ***reflection operators***. Table 1 shows the standard matrices for the reflections about the coordinate axes in $R^2$, and Table 2 shows the standard matrices for the reflections about the coordinate planes in $R^3$. In each case the standard matrix was obtained using the following procedure introduced in Section 1.8: Find the images of the standard basis vectors, convert those images to column vectors, and then use those column vectors as successive columns of the standard matrix.

Table 1

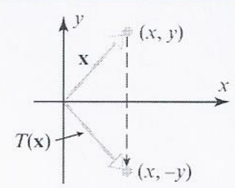
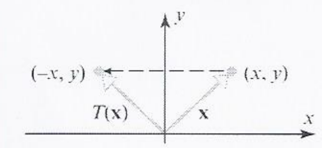
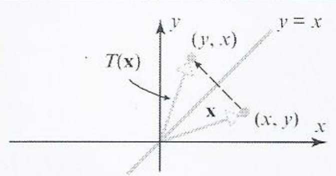
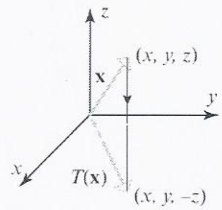
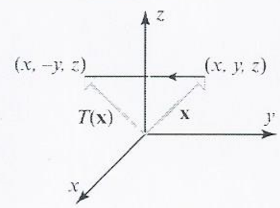
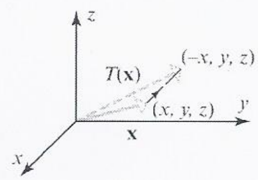
| Operator | Illustration | Images of $\mathbf{e}_1$ and $\mathbf{e}_2$ | Standard Matrix |
|---|---|---|---|
| Reflection about the $x$-axis<br>$T(x, y) = (x, -y)$ |  | $T(\mathbf{e}_1) = T(1, 0) = (1, 0)$<br>$T(\mathbf{e}_2) = T(0, 1) = (0, -1)$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection about the $y$-axis<br>$T(x, y) = (-x, y)$ |  | $T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$<br>$T(\mathbf{e}_2) = T(0, 1) = (0, 1)$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection about the line $y = x$<br>$T(x, y) = (y, x)$ |  | $T(\mathbf{e}_1) = T(1, 0) = (0, 1)$<br>$T(\mathbf{e}_2) = T(0, 1) = (1, 0)$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |

Table 2

| Operator | Illustration | Images of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ | Standard Matrix |
|---|---|---|---|
| Reflection about the $xy$-plane<br>$T(x, y, z) = (x, y, -z)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ |
| Reflection about the $xz$-plane<br>$T(x, y, z) = (x, -y, z)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Reflection about the $yz$-plane<br>$T(x, y, z) = (-x, y, z)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$ | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

Projection Operators

Matrix operators on $R^2$ and $R^3$ that map each point into its orthogonal projection onto a fixed line or plane through the origin are called ***projection operators*** (or more precisely, ***orthogonal projection*** operators). Table 3 shows the standard matrices for the orthogonal projections onto the coordinate axes in $R^2$, and Table 4 shows the standard matrices for the orthogonal projections onto the coordinate planes in $R^3$.

Table 3

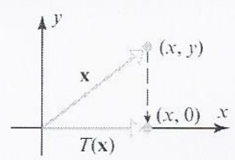
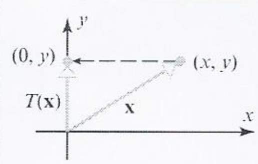
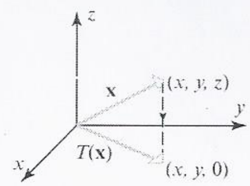
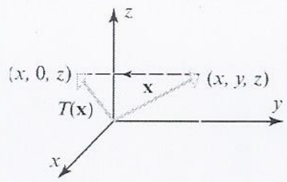
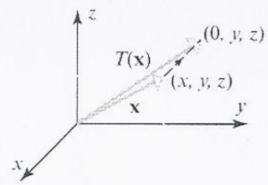
| Operator | Illustration | Images of $\mathbf{e}_1$ and $\mathbf{e}_2$ | Standard Matrix |
|---|---|---|---|
| Orthogonal projection onto the $x$-axis<br>$T(x, y) = (x, 0)$ |  | $T(\mathbf{e}_1) = T(1, 0) = (1, 0)$<br>$T(\mathbf{e}_2) = T(0, 1) = (0, 0)$ | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ |
| Orthogonal projection onto the $y$-axis<br>$T(x, y) = (0, y)$ |  | $T(\mathbf{e}_1) = T(1, 0) = (0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1) = (0, 1)$ | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |

Table 4

| Operator | Illustration | Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ | Standard Matrix |
|---|---|---|---|
| Orthogonal projection onto the $xy$-plane<br>$T(x, y, z) = (x, y, 0)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| Orthogonal projection onto the $xz$-plane<br>$T(x, y, z) = (x, 0, z)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Orthogonal projection onto the $yz$-plane<br>$T(x, y, z) = (0, y, z)$ |  | $T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$<br>$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$<br>$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

**Rotation Operators** Matrix operators on $R^2$ and $R^3$ that move points along arcs of circles centered at the origin are called ***rotation operators***. Let us consider how to find the standard matrix for the rotation operator $T: R^2 \to R^2$ that moves points *counterclockwise* about the origin through a positive angle $\theta$. As illustrated in Figure 4.9.1, the images of the standard basis vectors are

$$T(\mathbf{e}_1) = T(1, 0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0, 1) = (-\sin\theta, \cos\theta)$$

so it follows from Formula (14) of Section 1.8 that the standard matrix for $T$ is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

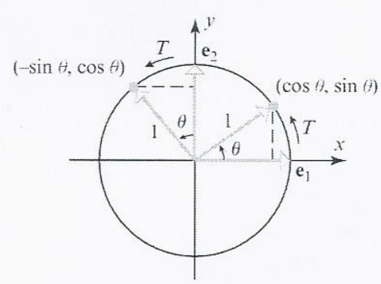



Figure 4.9.1

In keeping with common usage we will denote this operator by $R_\theta$ and call

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

In the plane, counterclockwise angles are positive and clockwise angles are negative. The rotation matrix for a *clockwise* rotation of $-\theta$ radians can be obtained by replacing $\theta$ by $-\theta$ in (1). After simplification this yields

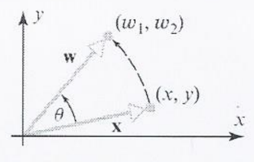
$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

the ***rotation matrix*** for $R^2$. If $\mathbf{x} = (x, y)$ is a vector in $R^2$, and if $\mathbf{w} = (w_1, w_2)$ is its image under the rotation, then the relationship $\mathbf{w} = R_\theta \mathbf{x}$ can be written in component form as

$$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned} \tag{2}$$

These are called the ***rotation equations*** for $R^2$. These ideas are summarized in Table 5.

Table 5

| Operator | Illustration | Rotation Equations | Standard Matrix |
|---|---|---|---|
| Counterclockwise rotation about the origin through an angle $\theta$ |   | $w_1 = x\cos\theta - y\sin\theta$<br>$w_2 = x\sin\theta + y\cos\theta$ | $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ |

**EXAMPLE 1 A Rotation Operator**

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ radians ($= 30°$) about the origin.

*Solution* It follows from (1) with $\theta = \pi/6$ that

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

or in comma-delimited notation, $R_{\pi/6}(1, 1) \approx (0.37, 1.37)$.

**Rotations in $R^3$**

A rotation of vectors in $R^3$ is commonly described in relation to a line through the origin called the ***axis of rotation*** and a unit vector $\mathbf{u}$ along that line (Figure 4.9.2$a$). The unit vector and what is called the ***right-hand rule*** can be used to establish a sign for the angle of rotation by cupping the fingers of your right hand so they curl in the direction of rotation and observing the direction of your thumb. If your thumb points in the direction of $\mathbf{u}$, then the angle of rotation is regarded to be ***positive*** relative to $\mathbf{u}$, and if it points in the direction opposite to $\mathbf{u}$, then it is regarded to be ***negative*** relative to $\mathbf{u}$ (Figure 4.9.2$b$).

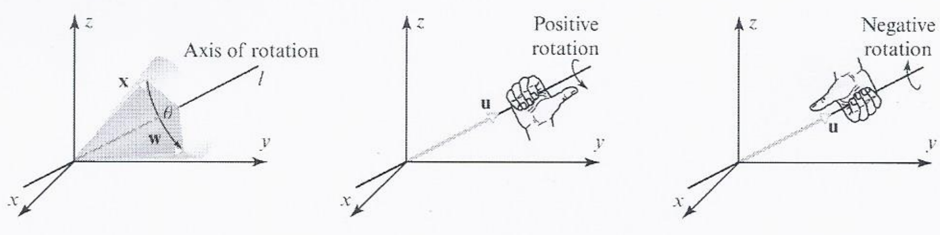
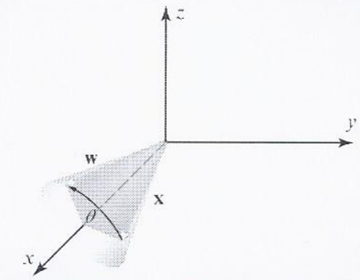
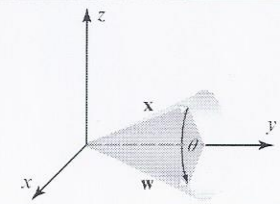
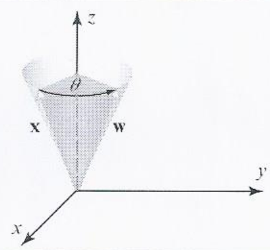



Figure 4.9.2 ($a$) Angle of rotation ($b$) Right-hand rule

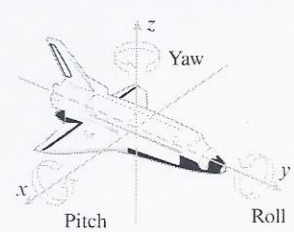
For rotations about the coordinate axes in $R^3$, we will take the unit vectors to be $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, in which case an angle of rotation will be positive if it is counterclockwise looking toward the origin along the positive coordinate axis and will be negative if it is clockwise. Table 6 shows the standard matrices for the ***rotation operators*** on $R^3$ that rotate each vector about one of the coordinate axes through an angle $\theta$. You will find it instructive to compare these matrices to that in Table 5.

Table 6

| Operator | Illustration | Rotation Equations | Standard Matrix |
|---|---|---|---|
| Counterclockwise rotation about the positive $x$-axis through an angle $\theta$ |   | $w_1 = x$ <br> $w_2 = y\cos\theta - z\sin\theta$ <br> $w_3 = y\sin\theta + z\cos\theta$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ |
| Counterclockwise rotation about the positive $y$-axis through an angle $\theta$ |   | $w_1 = x\cos\theta + z\sin\theta$ <br> $w_2 = y$ <br> $w_3 = -x\sin\theta + z\cos\theta$ | $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ |
| Counterclockwise rotation about the positive $z$-axis through an angle $\theta$ |   | $w_1 = x\cos\theta - y\sin\theta$ <br> $w_2 = x\sin\theta + y\cos\theta$ <br> $w_3 = z$ | $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

### Yaw, Pitch, and Roll

In aeronautics and astronautics, the orientation of an aircraft or space shuttle relative to an $xyz$-coordinate system is often described in terms of angles called ***yaw***, ***pitch***, and ***roll***. If, for example, an aircraft is flying along the $y$-axis and the $xy$-plane defines the horizontal, then the aircraft's angle of rotation about the $z$-axis is called the ***yaw***, its angle of rotation about the $x$-axis is called the ***pitch***, and its angle of rotation about the $y$-axis is called the ***roll***. A combination of yaw, pitch, and roll can be achieved by a single rotation about some axis through the origin. This is, in fact, how a space shuttle makes attitude adjustments—it doesn't perform each rotation separately; it calculates one axis, and rotates about that axis to get the correct orientation. Such rotation maneuvers are used to align an antenna, point the nose toward a celestial object, or position a payload bay for docking.




For completeness, we note that the standard matrix for a counterclockwise rotation through an angle $\theta$ about an axis in $R^3$, which is determined by an arbitrary *unit vector* $\mathbf{u} = (a, b, c)$ that has its initial point at the origin, is

$$\begin{bmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{3}$$

The derivation can be found in the book *Principles of Interactive Computer Graphics*, by W. M. Newman and R. F. Sproull (New York: McGraw-Hill, 1979). You may find it instructive to derive the results in Table 6 as special cases of this more general result.

**Dilations and Contractions** If $k$ is a nonnegative scalar, then the operator $T(\mathbf{x}) = k\mathbf{x}$ on $R^2$ or $R^3$ has the effect of increasing or decreasing the length of each vector by a factor of $k$. If $0 \le k < 1$ the operator is called a ***contraction*** with factor $k$, and if $k > 1$ it is called a ***dilation*** with factor $k$ (Figure 4.9.3). Tables 7 and 8 illustrate these operators. If $k = 1$, then $T$ is the identity operator.


Figure 4.9.3 (a) $0 \le k < 1$ (b) $k > 1$

Not required to memorize, but be aware of

Table 7

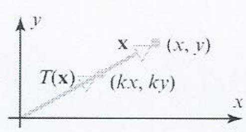
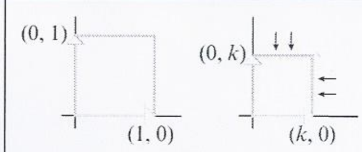
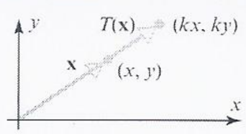
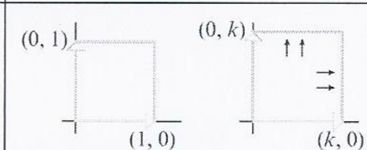
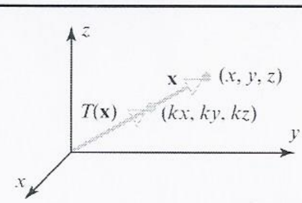
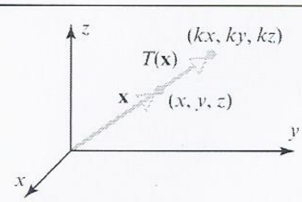
| Operator | Illustration $T(x, y) = (kx, ky)$ | Effect on the Unit Square | Standard Matrix |
|---|---|---|---|
| Contraction with factor $k$ in $R^2$ $(0 \le k < 1)$ |   |   | $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ |
| Dilation with factor $k$ in $R^2$ $(k > 1)$ |   |   | |

Table 8

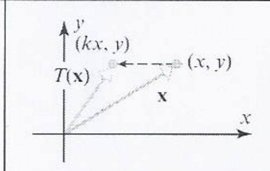
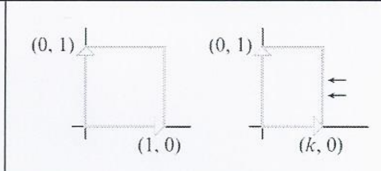
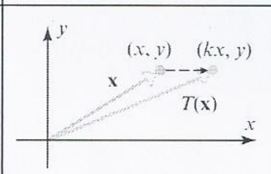
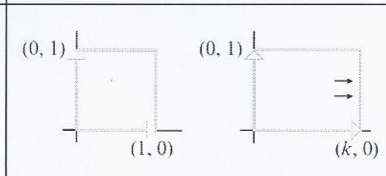
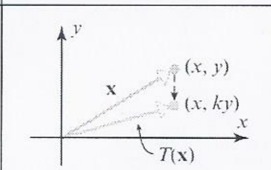
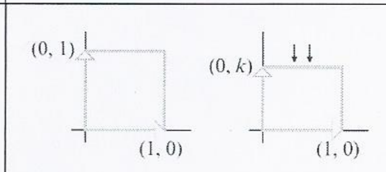
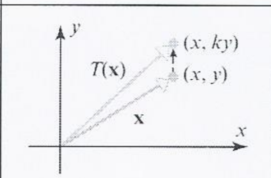
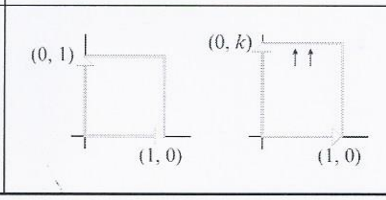
| Operator | Illustration $T(x, y, z) = (kx, ky, kz)$ | Standard Matrix |
|---|---|---|
| Contraction with factor $k$ in $R^3$ $(0 \le k < 1)$ |   | $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ |
| Dilation with factor $k$ in $R^3$ $(k > 1)$ |   | |

**Expansions and Compressions**

In a dilation or contraction of $R^2$ or $R^3$, *all* coordinates are multiplied by a nonnegative factor $k$. If only *one* coordinate is multiplied by $k$, then, depending on the value of $k$, the resulting operator is called a ***compression*** or ***expansion*** with factor $k$ in the direction of a coordinate axis. This is illustrated in Table 9 for $R^2$. The extension to $R^3$ is left as an exercise.

Table 9

Not required to memorize, but be aware of

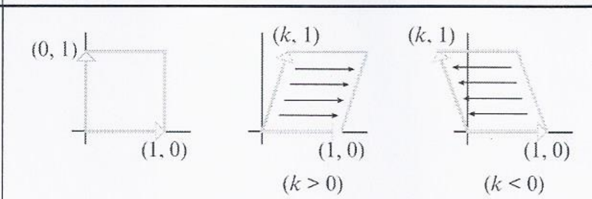
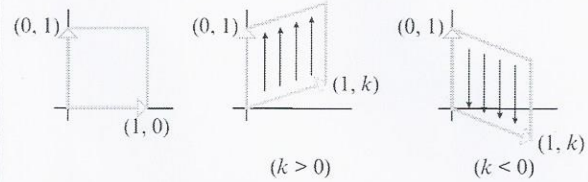
| Operator | Illustration $T(x, y) = (kx, y)$ | Effect on the Unit Square | Standard Matrix |
|---|---|---|---|
| Compression in the $x$-direction with factor $k$ in $R^2$ $(0 \le k < 1)$ |  |  | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| Expansion in the $x$-direction with factor $k$ in $R^2$ $(k > 1)$ |  |  | |
| **Operator** | **Illustration $T(x, y) = (x, ky)$** | **Effect on the Unit Square** | **Standard Matrix** |
| Compression in the $y$-direction with factor $k$ in $R^2$ $(0 \le k < 1)$ |  |  | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |
| Expansion in the $y$-direction with factor $k$ in $R^2$ $(k > 1)$ |  |  | |

**Shears**

A matrix operator of the form $T(x, y) = (x + ky, y)$ translates a point $(x, y)$ in the $xy$-plane parallel to the $x$-axis by an amount $ky$ that is proportional to the $y$-coordinate of the point. This operator leaves the points on the $x$-axis fixed (since $y = 0$), but as we progress away from the $x$-axis, the translation distance increases. We call this operator the ***shear in the x-direction by a factor k***. Similarly, a matrix operator of the form $T(x, y) = (x, y + kx)$ is called the ***shear in the y-direction by a factor k***. Table 10, which illustrates the basic information about shears in $R^2$, shows that a shear is in the positive direction if $k > 0$ and the negative direction if $k < 0$.

Not required to memorize, but be aware of

Table 10

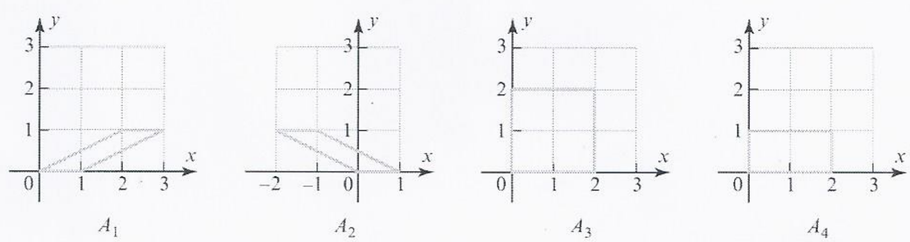
| Operator | Effect on the Unit Square | Standard Matrix |
|---|---|---|
| Shear in the $x$-direction by a factor $k$ in $R^2$ $T(x, y) = (x + ky, y)$ |   | $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ |
| Shear in the $y$-direction by a factor $k$ in $R^2$ $T(x, y) = (x, y + kx)$ |   | $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ |

**EXAMPLE 2 Effect of Matrix Operators on the Unit Square**

In each part, describe the matrix operator whose standard matrix is shown, and show its effect on the unit square.

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ (c) $A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

*Solution* By comparing the forms of these matrices to those in Tables 7, 9, and 10, we see that the matrix $A_1$ corresponds to a shear in the $x$-direction by a factor 2, the matrix $A_2$ corresponds to a shear in the $x$-direction by a factor $-2$, the matrix $A_3$ corresponds to a dilation with factor 2, and the matrix $A_4$ corresponds to an expansion in the $x$-direction with factor 2. The effects of these operators on the unit square are shown in Figure 4.9.4. ◀

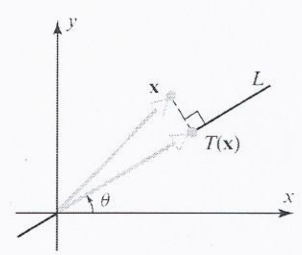



▶ Figure 4.9.4

**Orthogonal Projections onto Lines Through the Origin**

In Table 3 we listed the standard matrices for the orthogonal projections onto the coordinate axes in $R^2$. These are special cases of the more general matrix operator $T_A: R^2 \to R^2$ that maps each point into its orthogonal projection onto a line $L$ through the origin that makes an angle $\theta$ with the positive $x$-axis (Figure 4.9.5). In Example 4 of Section 3.3 we used Formula (10) of that section to find the orthogonal projections of the standard basis vectors for $R^2$ onto that line. Expressed in matrix form, we found those projections to be

$$T(\mathbf{e}_1) = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$




▲ Figure 4.9.5

Thus, the standard matrix for $T_A$ is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$