Math 392 Test 4B

May 15, 2018

Name: SOLUTIONS

Note that both sides of each page may have printed material.

Instructions:

1. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.

2. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.

3. Write neatly so that I am able to follow your sequence of steps and box your answers.

4. Read through the exam and complete the problems that are easy (for you) first!

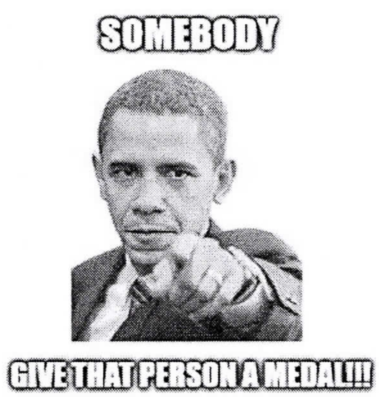
5. No calculators, notes or other aids allowed! Including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**

6. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

7. Don't commit any of the blasphemies mentioned in the syllabus!

8. Other than that, have fun and good luck!

You survived to the end of Math 392???



1. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

(a) (20 points) Find the inverse of A.

(b) (10 points) Use your answer to part (a) to solve the system

$$\begin{aligned} x + 2y + 2z &= 1 \\ x \phantom{{}+2y} + z &= 0 \\ x + 2y + z &= 1 \end{aligned}$$

(a) By Row Reduction

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} R_2 \\ R_3 - R_2 \\ R_1 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2/2 \\ R_3 \end{array}$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ 1 & 0 & -1 \end{pmatrix}}$$

OR

By Adjoint Method

$$\det A = \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{array}{l} R_1 - R_3 \\ R_2 \\ R_3 \end{array}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$$

$$\mathrm{Adj}(A) = \begin{pmatrix} \oplus\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \ominus\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \oplus\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ \ominus\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \oplus\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \ominus\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ \oplus\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} & \ominus\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \oplus\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -2 & 0 & 2 \\ 2 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}^T = \begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \mathrm{Adj}(A) = \frac{1}{2} \begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ 1 & 0 & -1 \end{pmatrix}}$$

(b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}}$$

Check:

$0 + 2(\tfrac{1}{2}) + 2(0) = 1$ ✓

$0 + (0) = 0$ ✓

$0 + 2(\tfrac{1}{2}) + (0) = 1$ ✓

2. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

(a) (20 points) Compute: (i) $\det A$, (ii) $\det A^{-1}$, (iii) $\det(2A^3 A^T A^{-2})$.

(b) (10 points) Use Cramer's Rule to solve for x only (do not solve for y or z!!!) in the following system. No credit for any other method.

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & 1 \\ x & & & + & z & = & 0 \\ x & + & 2y & + & z & = & 1 \end{array}$$

(a) (i) $\boxed{\det A = 2}$

(See problem 1(a))

$\Rightarrow$ (ii) $\det A^{-1} = \frac{1}{\det A} = \boxed{\frac{1}{2}}$

$\Rightarrow$ (iii) $\det(2A^3 A^T A^{-2}) = 2^3 (\det A)^3 (\det A)(\det A)^{-2}$

$= 2^3 \cdot 2^3 \cdot 2 \cdot 2^{-2}$

$= 2^5$

$= \boxed{32}$

(b) $D = \det A = 2$

$D_x = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= 0$

$\Rightarrow x = \frac{D_x}{D}$

$= \frac{0}{2}$

$\Rightarrow \boxed{x = 0} \longrightarrow$ confirmed by 1(b)

3. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$.

(a) (30 points) Find the eigenvalues and corresponding eigenvectors of A. Indicate which vectors correspond with which values using appropriate notation.

(a) Eigenvalues

↳ Solve $\det(\lambda I - A) = 0$

$$\Rightarrow \begin{vmatrix} \lambda-2 & 1 \\ -2 & \lambda-5 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)(\lambda-5)+2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 12 = 0$$

$$\Rightarrow (\lambda-4)(\lambda-3) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 4 \ , \ \lambda_2 = 3}$$

↳ Eigenvalues.

Eigenvectors

↳ Solve $(\lambda_i I - A)\vec{x} = \vec{0}$

$\lambda_1 = 4$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_1+R_2 \end{array}$$

$\Rightarrow x_2 = t$

$\Rightarrow x_1 = -\frac{1}{2}t$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \equiv t_1 \boxed{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

$\vec{\lambda}_1$ = eigenvector corresponding to $\lambda_1 = 4$.

$\lambda_2 = 3$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ 2R_1+R_2 \end{array}$$

$\Rightarrow x_2 = t$

$\Rightarrow x_1 = -t$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$\vec{\lambda}_2$ = eigenvector corresponding to $\lambda_2 = 3$.

(b) (10 points) Use your answer in part (a) to find the general solution to the system

$$\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 2y_1 + 5y_2 \end{aligned}$$

From 3, (a): Eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 3$

Eigenvectors are $\vec{\lambda}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\vec{\lambda}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow$ General soln: $\boxed{\vec{y} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}}$

By the formula

$\vec{y} = c_1 \vec{\lambda}_1 e^{\lambda_1 t} + c_2 \vec{\lambda}_2 e^{\lambda_2 t}$

This problem also represents the easiest 10 points this whole semester. So that's neat.

Bonus Problems: Note that you must attempt all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.

1. Let A be the matrix in problem 3. Diagonalize A by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (5 points).

By formulas in class: $P = (\vec{\lambda}_1 | \vec{\lambda}_2)$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\Rightarrow$ By 3, (a): $P = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

2. Let A be as in problem 1 above. Compute A^6 as a 2x2 matrix (5 points).

$$A^6 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4^6 & 0 \\ 0 & 3^6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4^6 & -3^6 \\ 2\cdot 4^6 & 3^6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4^6 + 2\cdot 3^6 & -4^6 + 3^6 \\ 2\cdot 4^6 - 2\cdot 3^6 & 2\cdot 4^6 - 3^6 \end{pmatrix}$$

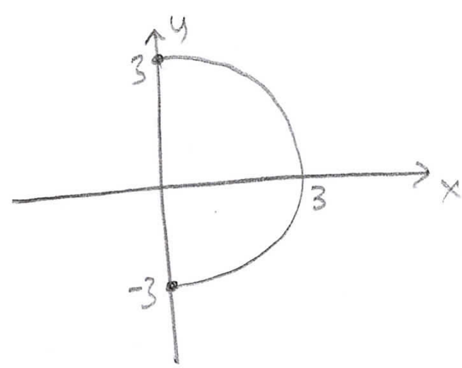
3. A thin wire is bent into the shape of a semi-circle $x^2 + y^2 = 9$, $x \geq 0$. If the linear density is given by $\rho(x, y) = y + 1$, find the mass of the wire (5 points).

$$\text{mass} = m = \int_C y + 1 \, ds$$

$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$\Rightarrow \vec{r}\,'(t) = \langle -3\sin t, 3\cos t \rangle$

$\Rightarrow |\vec{r}\,'(t)| = 3$



$$\Rightarrow m = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\sin t + 1)\, 3\, dt$$

$$= 3\left(-3\cos t + t\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= 3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \boxed{3\pi}$$

4. Let $\vec{F} = < 4y, 2x, 4y - 4x >$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the circle of radius 4 in the plane $x + y + z = 4$ centered at (4,4,-4) and oriented clockwise when viewed from the origin (5 points).

By Stokes' Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$

$$\text{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2x & 4y-4x \end{vmatrix}$$

$$= \langle 4, 4, -2 \rangle$$

$$\vec{n} = \langle 1, 1, 1 \rangle$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \iint_S \langle 4,4,-2 \rangle \cdot \frac{\langle 1,1,1 \rangle}{\sqrt{3}} \, dS$$

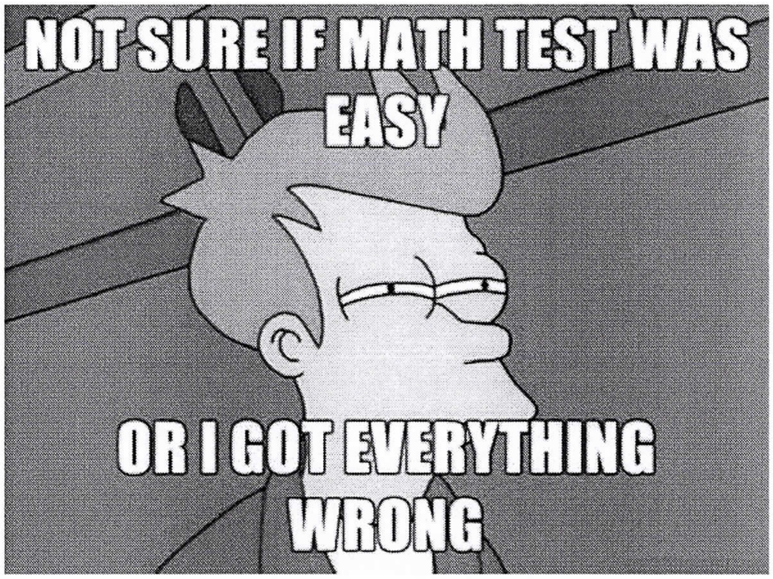
$$= \frac{6}{\sqrt{3}} \iint_S dS$$

$$= \frac{6}{\sqrt{3}} \cdot \pi (4)^2$$

$$= \boxed{6 \cdot \frac{16\pi}{\sqrt{3}}}$$

OR

$$= \boxed{\frac{96\pi}{\sqrt{3}}}$$



Jhevon: *SLAP* ALWAYS believe in yourself! No matter what!

Me knowing Jhevon speaks truth: