

Name: SOLUTIONS

Note that both sides of each page may have printed material.

**Instructions:**

1. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
2. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
3. Write neatly so that I am able to follow your sequence of steps and box your answers.
4. Read through the exam and complete the problems that are easy (for you) first!
5. No calculators, notes or other aids allowed! Including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**
6. Use the correct notation and write what you mean!  $x^2$  and  $x2$  are not the same thing, for example, and I will grade accordingly.
7. Don't commit any of the blasphemies mentioned in the syllabus!
8. Other than that, have fun and good luck!
9. **IMPORTANT:** All closed surfaces on this exam are positively oriented!

When u finally start studying hard  
and Jhevon be like

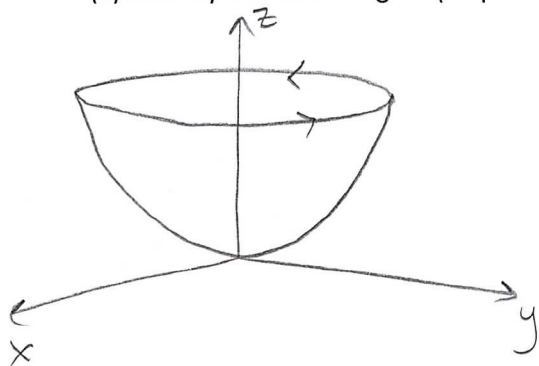


**NO MORE QUIZZES!!**

1. Let  $C$  be the curve of intersection of the paraboloid  $z = x^2 + y^2$  and the plane  $z = 1$ . Let  $\vec{F} = \langle 2y^2, 3x, z^2 \rangle$ .

Compute  $\oint_C \vec{F} \cdot d\vec{r}$  in two ways:

(a) Directly as a line integral. (20 points)



$$C: \vec{r}(t) = \langle \cos t, \sin t, 1 \rangle, 0 \leq t \leq 2\pi$$

$$d\vec{r} = \langle -\sin t, \cos t, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 2\sin^2 t, 3\cos t, 1 \rangle$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -2\sin^3 t + 3\cos^2 t \, dt$$

$$= \int_0^{2\pi} -2\sin t(1 - \cos^2 t) + \frac{3}{2} + \frac{3}{2} \cos 2t \, dt$$

$$= \int_0^{2\pi} -2\sin t + 2\sin t \cos^2 t + \frac{3}{2} \, dt$$

$$= -\frac{2}{3} \cos^3 t + \frac{3}{2} t \Big|_0^{2\pi}$$

$$= -\frac{2}{3} + \frac{3}{2}(2\pi) + \frac{2}{3}$$

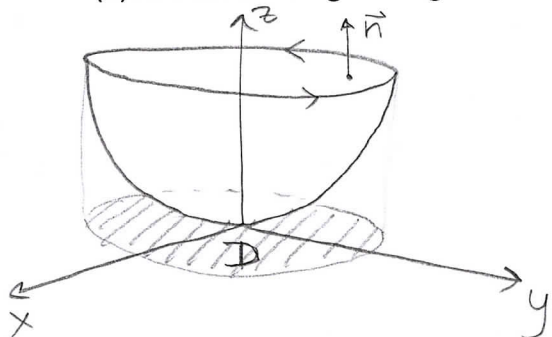
$$= \boxed{3\pi}$$

(This is problem 1 recopied for your convenience. Do part (b) on this page!)

Let  $C$  be the curve of intersection of the paraboloid  $z = x^2 + y^2$  and the plane  $z = 1$ . Let  $\vec{F} = \langle 2y^2, 3x, z^2 \rangle$ .

Compute  $\oint_C \vec{F} \cdot d\vec{r}$  in two ways:

(b) As a double integral using Stokes' Theorem. (20 points)



By Stokes' Theorem,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl} \vec{F} \cdot d\vec{S} \\ &= \iint_D \text{curl} \vec{F} \cdot \vec{n} \, dA\end{aligned}$$

Take  $S$  to be the plane  $z = 1$ .

$$\Rightarrow \vec{n} = \langle -f_x, -f_y, 1 \rangle$$

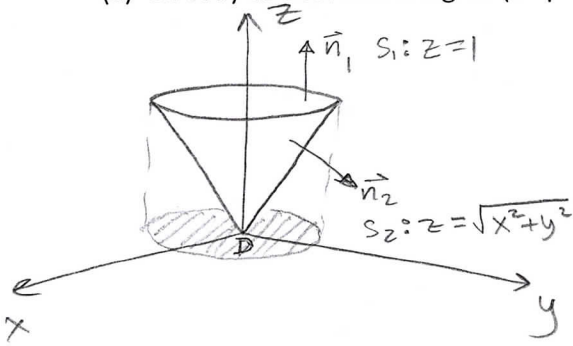
$= \langle 0, 0, 1 \rangle \rightarrow$  Obvious from the sketch. This could be stated on the exam without justification.

$$\begin{aligned}\text{Also, } \text{curl} \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x & z^2 \end{vmatrix} \\ &= \langle 0, 0, 3 - 4y \rangle\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_C \vec{F} \cdot d\vec{r} &= \iint_D \langle 0, 0, 3 - 4y \rangle \cdot \langle 0, 0, 1 \rangle \, dA \\ &= \iint_D 3 - 4y \, dA \\ &= \int_0^{2\pi} \int_0^1 (3 - 4r \sin \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 3r \, dr \, d\theta \\ &= \boxed{3\pi}\end{aligned}$$

2. Let  $E$  be the solid region bounded by the cone  $z = \sqrt{x^2 + y^2}$  and the plane  $z = 1$ . Let  $S$  be the two-part boundary of  $E$ . If  $\vec{F} = \langle x, y, z \rangle$ , compute the total flux over  $E$  in two ways:

(a) Directly as a surface integral. (20 points)



$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} dA$$

$$\underline{S_1}: \vec{n}_1 = \langle -f_x, -f_y, 1 \rangle, f = 1.$$

$$= \langle 0, 0, 1 \rangle$$

$$\Rightarrow \iint_{S_1} = \iint_D \langle x, y, z \rangle \cdot \langle 0, 0, 1 \rangle dA$$

$$= \iint_D z dA$$

$$= \iint_D 1 dA$$

$$= \text{area of } D$$

$$= \pi$$

$$\underline{S_2}: \vec{n}_2 = \langle f_x, f_y, -1 \rangle, f = \sqrt{x^2 + y^2}$$

$$= \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right\rangle$$

$$\Rightarrow \iint_{S_2} = \iint_D \langle x, y, z \rangle \cdot \vec{n} dA$$

$$= \iint_D \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - z dA$$

$$= \iint_D \sqrt{x^2 + y^2} - \sqrt{x^2 + y^2} dA$$

$$= \iint_D 0 dA$$

$$= 0$$

$$\Rightarrow \text{Flux} = \iint_{S_1} + \iint_{S_2}$$

$$= \boxed{\pi}$$

(This is problem 2 recopied for your convenience. Do part (b) on this page!)

Let  $E$  be the solid region bounded by the cone  $z = \sqrt{x^2 + y^2}$  and the plane  $z = 1$ . Let  $S$  be the two-part boundary of  $E$ . If  $\vec{F} = \langle x, y, z \rangle$ , compute the total flux over  $E$  in two ways:

(b) As a triple integral using the Divergence Theorem. (20 points)

By the Divergence Theorem,

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$$

$$= \iiint_E 3 \, dV \quad (\text{since } \text{div } \vec{F} = 3)$$

$$= \int_0^{2\pi} \int_0^1 \int_r^1 3r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 3r(1-r) \, dr$$

$$= 6\pi \left[ \frac{r^2}{2} - \frac{r^3}{3} \right] \Big|_0^1$$

$$= 6\pi \left( \frac{1}{2} - \frac{1}{3} \right)$$

$$= 6\pi \left( \frac{1}{6} \right)$$

$$= \boxed{\pi}$$

3. (a) (15 points) Find the reduced row-echelon form of:

$$\begin{pmatrix} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 1 & -5 & 6 \end{pmatrix} \begin{array}{l} R_1 \\ 2R_1 - R_2 \\ 3R_1 - R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - R_2 \\ R_2 \\ R_2 - R_3 \end{array}$$

→ RREF

(b) (5 points) Hence, find the solution to the system:  $\begin{cases} x + y - 3z = 4 \\ 2x + y - z = 2 \\ 3x + 2y - 4z = 6 \end{cases}$ . Write your solution as a sum of column vectors if applicable.

From above:  $z = t$

$$R_2: y - 5t = 6$$

$$\Rightarrow y = 5t + 6$$

$$R_1: x + 2t = -2$$

$$\Rightarrow x = -2t - 2$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2t - 2 \\ 5t + 6 \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$$

**Bonus Problems:** Note that you must attempt all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.

1. Let  $A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -1 \\ 2 & 3 & 1 \end{pmatrix}$ . Find  $\det A$  and  $\det(3A^3A^T)$  (10 points).

$$\det A = \det \begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -1 \\ 0 & -3 & -1 \end{pmatrix} \Big|_{R_2 - 2R_1} = \begin{vmatrix} -4 & -1 \\ -3 & -1 \end{vmatrix} = \boxed{1}$$

$$\begin{aligned} \Rightarrow \det(3A^3A^T) &= 3^3 (\det A)^3 (\det A) \\ &= 27 (1)^3 (1) \\ &= \boxed{27} \end{aligned}$$

2. For  $A$  as above, find  $A^{-1}$  (5 points).

Row Reduction

$$\left( \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array} \right) \begin{array}{l} R_1 \\ R_2 \\ 2R_1 - R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -12 & -3 & 0 & 3 & 0 \\ 0 & 12 & 4 & 8 & 0 & -4 \end{array} \right) \begin{array}{l} R_1 - R_3 \\ R_2 \times 3 \\ R_3 \times 4 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -4 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 8 & 3 & -4 \end{array} \right) \begin{array}{l} R_1 \\ R_2/3 \\ R_2 + R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -4 & 0 & 8 & 4 & -4 \\ 0 & 0 & 1 & 8 & 3 & -4 \end{array} \right) \begin{array}{l} R_1 \\ R_2 + R_3 \\ R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 8 & 3 & -4 \end{array} \right) \begin{array}{l} R_1 \\ R_2 / -4 \\ R_3 \end{array} \Rightarrow A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 1 \\ 8 & 3 & -4 \end{pmatrix}$$

OR

Adjoint Method

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A)$$

$$= \frac{1}{1} \begin{pmatrix} \oplus \begin{vmatrix} -4 & -1 \\ 3 & 1 \end{vmatrix} & \oplus \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} & \oplus \begin{vmatrix} 0 & -4 \\ 2 & 3 \end{vmatrix} \\ \oplus \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} & \oplus \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \oplus \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} \\ \oplus \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix} & \oplus \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \oplus \begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -1 & -2 & 8 \\ 0 & -1 & 3 \\ -1 & 1 & -4 \end{pmatrix}^T$$

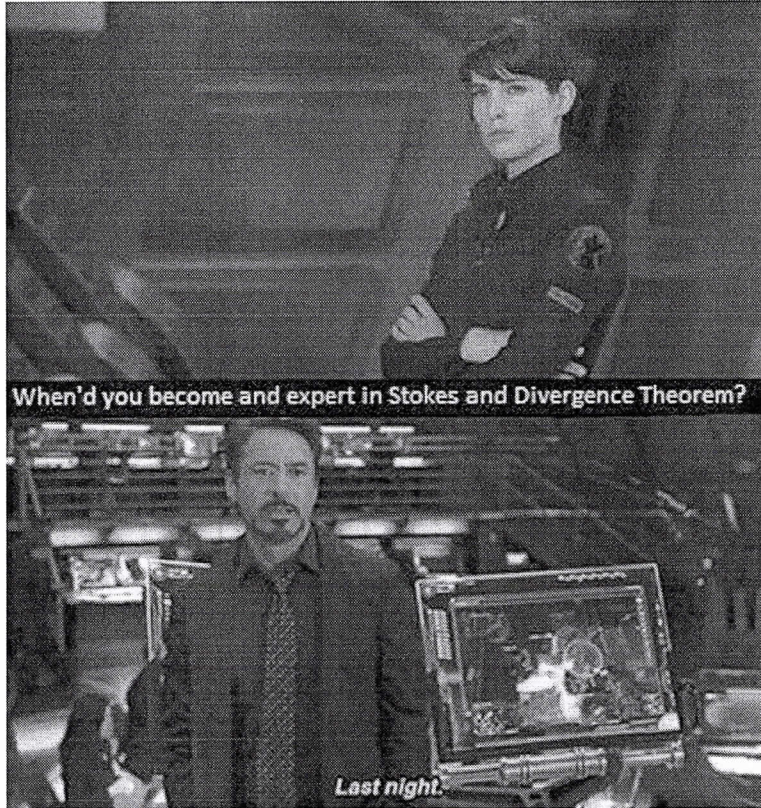
$$\Rightarrow A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 1 \\ 8 & 3 & -4 \end{pmatrix}$$

3. Using your answer to problem 2, solve the system:  $\begin{cases} x + 3y + z = 4 \\ \phantom{x} - 4y - z = -1 \\ 2x + 3y + z = 3 \end{cases}$  (5 points).

$$A\vec{x} = \vec{b} \text{ and } A^{-1} \text{ exists}$$

$$\Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & -1 & 1 \\ 8 & 3 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 17 \end{pmatrix}$$



**Question:** You saw Infinity War?  
**Only correct answer:** No, Jhevon; I was studying for your class. It's all I do.