

Name: SOLUTIONS

Note that both sides of each page may have printed material.

If you could read the directions
before asking me a question



Instructions:

1. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
2. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
3. Write neatly so that I am able to follow your sequence of steps and box your answers.
4. Read through the exam and complete the problems that are easy (for you) first!
5. No calculators, notes or other aids allowed! Including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**
6. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
7. Don't commit any of the blasphemies mentioned in the syllabus!
8. Other than that, have fun and good luck!

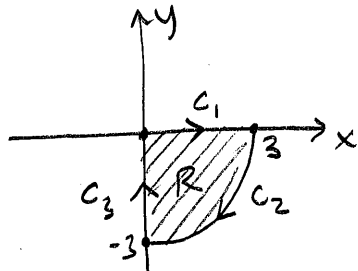
AND THEN I SAID...



1. Let R be the region in the fourth quadrant bounded by $x^2 + y^2 = 9$, $y = 0$, and $x = 0$. Let C be the boundary curve of R , oriented clockwise.

Compute $\oint_C \left(9 - \frac{y}{3}\right) dx - 4 dy$ in two ways:

(a) Directly as a line integral. (10 points)



$$\underline{C_1}: x = 3t, y = 0, 0 \leq t \leq 1$$

$$dx = 3dt, dy = 0dt$$

$$\Rightarrow \int_{C_1} = \int_0^1 9 \cdot 3 dt$$

$$= 27$$

$$\underline{C_2}: x = 3\cos t, y = 3\sin t, t: 0 \rightarrow -\frac{\pi}{2}$$

$$dx = -3\sin t dt, dy = 3\cos t dt$$

$$\Rightarrow \int_{C_2} = \int_0^{-\frac{\pi}{2}} \left(9 - \sin t\right) (-3\sin t) - 4(3\cos t) dt$$

$$= \int_0^{-\frac{\pi}{2}} -27\sin t + 3\sin^2 t - 12\cos t dt$$

$$= \int_0^{-\frac{\pi}{2}} -27\sin t + \frac{3}{2} - \frac{3}{2}\cos 2t - 12\cos t dt$$

$$= 27\cos t + \frac{3}{2}t - \frac{3}{4}\sin 2t - 12\sin t \Big|_0^{-\frac{\pi}{2}}$$

$$= -\frac{3\pi}{4} + 12 - 27$$

$$= -\frac{3\pi}{4} - 15$$

$$\underline{C_3}: x = 0, y = -3 + 3t, 0 \leq t \leq 1$$

$$dx = 0dt, dy = 3dt$$

$$\Rightarrow \int_{C_3} = \int_0^1 -4(3) dt$$

$$= \int_0^1 -12 dt$$

$$= -12$$

$$\therefore \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= 27 - \frac{3\pi}{4} - 15 - 12$$

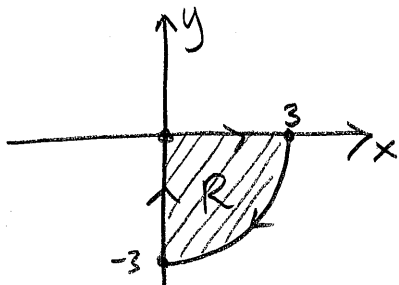
$$= -\frac{3\pi}{4}$$

(This is problem 1 recopied for your convenience. Do part (b) on this page!)

Let R be the region in the fourth quadrant bounded by $x^2 + y^2 = 9$, $y = 0$, and $x = 0$. Let C be the boundary curve of R , oriented *clockwise*.

Compute $\oint_C \left(9 - \frac{y}{3}\right) dx - 4 dy$ in two ways:

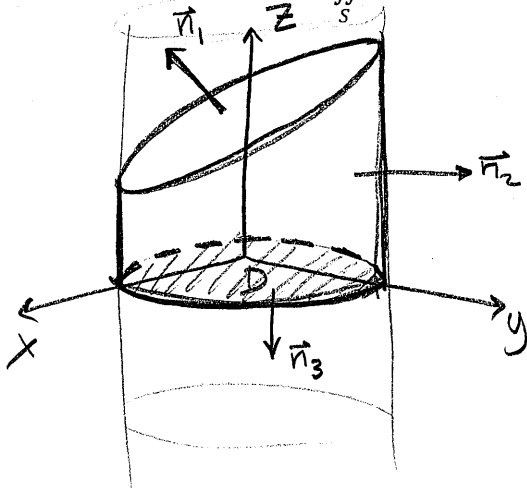
(b) As a double integral using Green's Theorem. (10 points)



$$\begin{aligned} \int_C \left(9 - \frac{y}{3}\right) dx - 4 dy &= - \iint_R Q_x - P_y dA \\ &= - \iint_R \left(0 - \left(-\frac{1}{3}\right)\right) dA \\ &= -\frac{1}{3} \iint_R dA \\ &= -\frac{1}{3} \cdot \frac{1}{4} \cdot \pi (3)^2 \\ &= \boxed{-\frac{3\pi}{4}} \end{aligned}$$

2. Let E be the solid region bounded by $x^2 + y^2 = 1$, $z = 0$, and $z + 3x = 9$. Let S be the positively oriented, three-part boundary of E . Let $\vec{F} = \langle x, y, z \rangle$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$. (20 points)



S_1 : $z = 9 - 3x$

$\Rightarrow \vec{n}_1 = \langle -f_x, -f_y, 1 \rangle = \langle 3, 0, 1 \rangle$

$\Rightarrow \iint_{S_1} = \iint_D \langle x, y, z \rangle \cdot \langle 3, 0, 1 \rangle dA$

$= \iint_D 3x + z dA$

$= \iint_D 3x + (9 - 3x) dA$

$= 9 \iint_D dA$

$= 9\pi$

S_2 : $x^2 + y^2 = 1$

$\Rightarrow \vec{r}(u, v) = \langle \cos u, \sin u, v \rangle$

$0 \leq u \leq 2\pi, 0 \leq v \leq 9 - 3\cos u$

$\Rightarrow \vec{r}_u = \langle -\sin u, \cos u, 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\Rightarrow \vec{n}_2 = \langle \cos u, \sin u, 0 \rangle$

$\Rightarrow \iint_{S_2} = \iint_R \langle \cos u, \sin u, v \rangle \cdot \langle \cos u, \sin u, 0 \rangle dA$

$= \iint_R dA$

$= \int_0^{2\pi} \int_0^{9-3\cos u} dv du$

$= \int_0^{2\pi} 9 - 3\cos u du$

$= 18\pi$

S_3 : $z = 0$

$\Rightarrow \vec{n}_3 = \langle 0, 0, -1 \rangle$

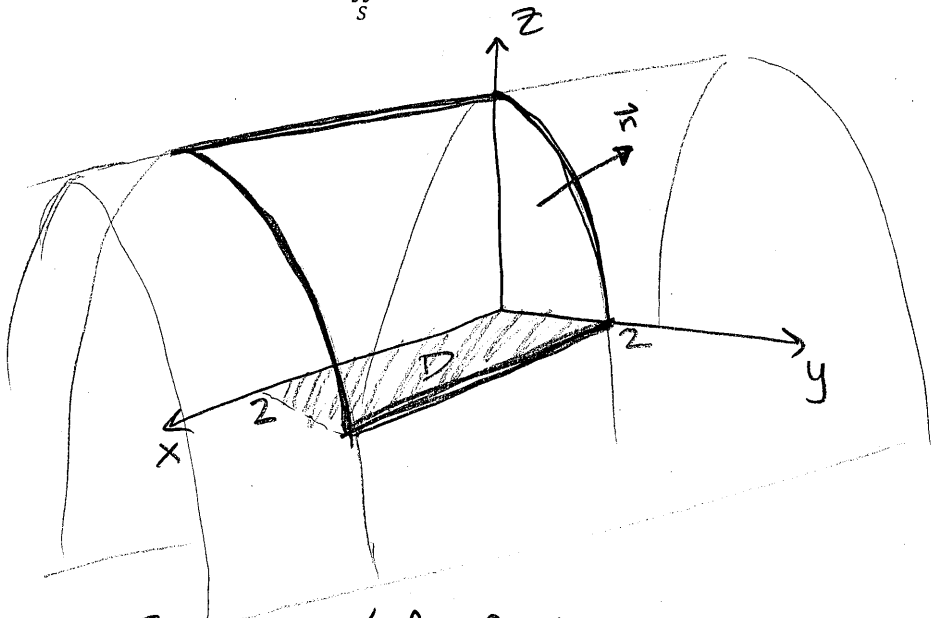
$\Rightarrow \iint_{S_3} = \iint_D \langle x, y, z \rangle \cdot \langle 0, 0, -1 \rangle dA$

$= \iint_D (-z) dA$

$= \iint_D 0 dA = 0$

$\Rightarrow \iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = 27\pi$

3. Compute $\iint_S y \, dS$, where S is the portion of $z = 4 - y^2$ in the first octant for $0 \leq x \leq 2$. (20 points)



$$\text{Set } \vec{n} = \langle -f_x, -f_y, 1 \rangle$$

$$\Rightarrow \vec{n} = \langle 0, 2y, 1 \rangle$$

$$\Rightarrow |\vec{n}| = \sqrt{4y^2 + 1}$$

$$\begin{aligned} \Rightarrow \iint_S &= \iint_D y \sqrt{4y^2 + 1} \, dA \\ &= \int_0^2 \int_0^2 y \sqrt{4y^2 + 1} \, dy \, dx \\ &= 2 \int_0^2 y \sqrt{4y^2 + 1} \, dy \end{aligned}$$

$$u^2 = 4y^2 + 1$$

$$\Rightarrow 2u \, du = 8y \, dy$$

$$\Rightarrow \frac{1}{4} u \, du = y \, dy$$

$$= \frac{2}{4} \int_1^{\sqrt{17}} u^2 \, du$$

$$= \frac{1}{2} \cdot \frac{u^3}{3} \Big|_1^{\sqrt{17}}$$

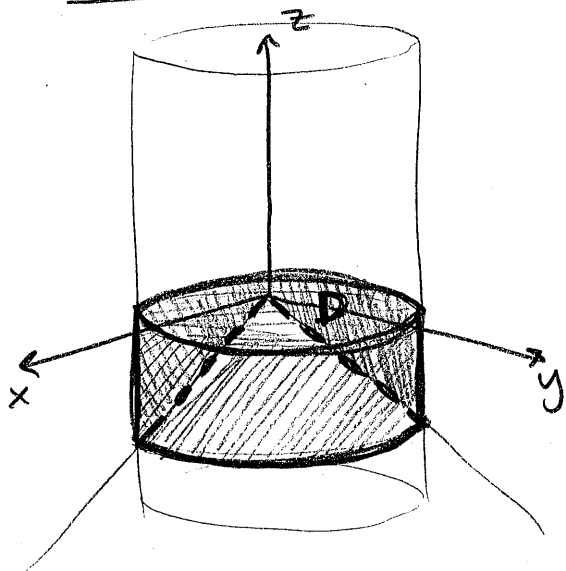
$$= \boxed{\frac{17^{3/2}}{6} - \frac{1}{6}}$$

4. Let T_1 be the region above the cone $z = -\sqrt{x^2 + y^2}$, inside the cylinder $x^2 + y^2 = 1$, below the xy -plane. Let T_2 be the region inside the sphere $x^2 + y^2 + z^2 = 1$, but above the cone $z = -\sqrt{x^2 + y^2}$.

(a) Set up two integrals: $\iiint_{T_1} (x^2 + y^2 + z^2) dV$ and $\iiint_{T_2} (x^2 + y^2 + z^2) dV$.

Use coordinate systems of your choice. (10 points)

To set up T_1



$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

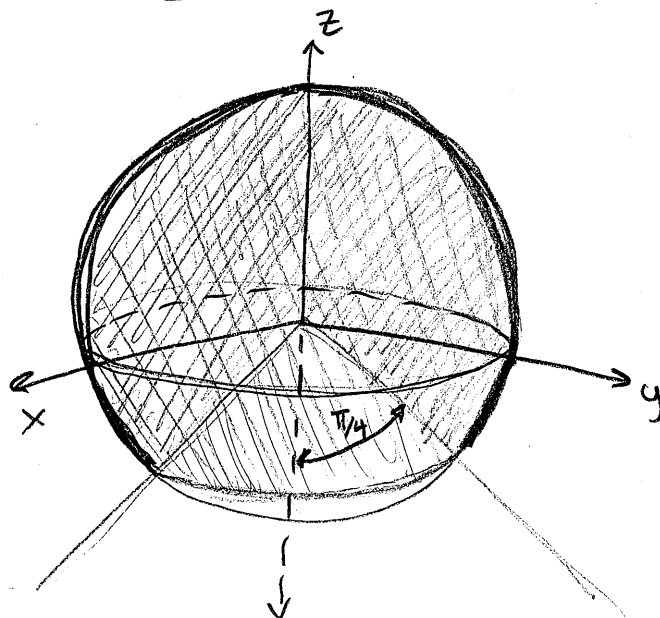
$$\text{bottom surf} \leq z \leq \text{top surf}$$

$$\Rightarrow -\sqrt{x^2 + y^2} = -r \leq z \leq 0$$

$$\Rightarrow \iiint_{T_1} = \int_0^{2\pi} \int_0^1 \int_{-r}^0 (r^2 + z^2) r dz dr d\theta$$

We used cylindrical coordinates.

To set up T_2



$$0 \leq \phi \leq \frac{3\pi}{4}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \rho \leq 1$$

$$\Rightarrow \iiint_{T_2} = \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} \int_0^1 \rho^2 \cdot \rho^2 \sin \phi d\rho d\phi d\theta$$

We used spherical coordinates.

(b) Evaluate either of the integrals set up in 4(a). (10 points)

If you chose to
evaluate \iiint_{T_1}

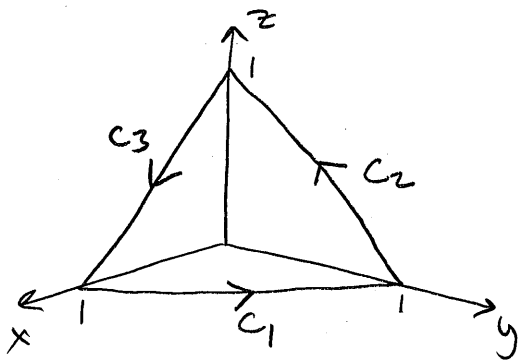
$$\begin{aligned}\iiint_{T_1} &= \int_0^{2\pi} \int_0^1 \int_{-r}^0 (r^2 + z^2) r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r^3 z + r \frac{z^3}{3} \Big|_{-r}^0 dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r^4 + \frac{r^4}{3} dr d\theta \\ &= 2\pi \int_0^1 \frac{4}{3} r^4 dr \\ &= 2\pi \left(\frac{4}{15} r^5 \right) \Big|_0^1 \\ &= \boxed{\frac{8\pi}{15}}\end{aligned}$$

If you chose to
evaluate \iiint_{T_2}

$$\begin{aligned}\iiint_{T_2} &= \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} \int_0^1 \rho^4 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} \frac{\rho^5}{5} \sin \phi \Big|_0^1 d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} \frac{1}{5} \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} -\frac{1}{5} \cos \phi \Big|_0^{\frac{3\pi}{4}} d\theta \\ &= 2\pi \left(\frac{1}{5} \cdot \frac{\sqrt{2}}{2} + \frac{1}{5} \right) \\ &= \boxed{\frac{\pi}{5} (\sqrt{2} + 2)}\end{aligned}$$

5. Let S be the part of the plane $x + y + z = 1$ in the first octant. Let C be the boundary of S oriented counter-clockwise when viewed from above. If $F = \langle x, y, xyz \rangle$, find $\int_C F \cdot dr$.

(20 points)



C_1 : line $(1,0,0) \rightarrow (0,1,0)$

$$\vec{r}(t) = \langle 1-t, t, 0 \rangle, 0 \leq t \leq 1$$

$$d\vec{r} = \langle -1, 1, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 1-t, t, 0 \rangle$$

$$\Rightarrow \int_{C_1} = \int_0^1 t - 1 + t \, dt$$

$$= \int_0^1 2t - 1 \, dt$$

$$= t^2 - t \Big|_0^1$$

$$= \boxed{0}$$

C_2 : line $(0,1,0) \rightarrow (0,0,1)$

$$\vec{r}(t) = \langle 0, 1-t, t \rangle, 0 \leq t \leq 1$$

$$d\vec{r} = \langle 0, -1, 1 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 0, 1-t, 0 \rangle$$

$$\Rightarrow \int_{C_2} = \int_0^1 t - 1 \, dt$$

$$= \frac{t^2}{2} - t \Big|_0^1$$

$$= \boxed{-\frac{1}{2}}$$

C_3 : line $(0,0,1) \rightarrow (1,0,0)$

$$\vec{r}(t) = \langle t, 0, 1-t \rangle, 0 \leq t \leq 1$$

$$d\vec{r} = \langle 1, 0, -1 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle t, 0, 0 \rangle$$

$$\Rightarrow \int_{C_3} = \int_0^1 t \, dt$$

$$= \frac{t^2}{2} \Big|_0^1$$

$$= \boxed{\frac{1}{2}}$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

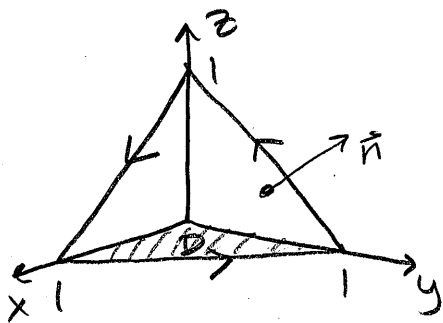
$$= 0 - \frac{1}{2} + \frac{1}{2}$$

$$= \boxed{0}$$

All that work for nothing!

Bonus Problems: Note that you must attempt all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.

1. Verify your answer to problem 5 by using Stokes' Theorem. (10 points)



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}$$

$$\text{curl} \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & xyz \end{vmatrix}$$

$$= \langle xz, -yz, 0 \rangle$$

$$z = 1 - x - y$$

$$\Rightarrow \vec{n} = \langle -f_x, -f_y, 1 \rangle$$

$$= \langle 1, 1, 1 \rangle$$

$$\Rightarrow \int_C = \iint_D \langle xz, -yz, 0 \rangle \cdot \langle 1, 1, 1 \rangle dA$$

$$= \int_0^1 \int_0^{1-x} xz - yz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x(1-x-y) - y(1-x-y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x - x^2 - xy - y + xy + y^2 \, dy \, dx$$

$$= \int_0^1 (x - x^2)y - \frac{y^2}{2} + \frac{y^3}{3} \Big|_0^{1-x} dx$$

$$= \int_0^1 (x - x^2)(1-x) - \frac{(1-x)^2}{2} + \frac{(1-x)^3}{3} dx$$

$$= \int_0^1 x(1-x)^2 - \frac{(1-x)^2}{2} + \frac{(1-x)^3}{3} dx$$

$$u = 1-x, \Rightarrow x = 1-u$$

$$du = -dx$$

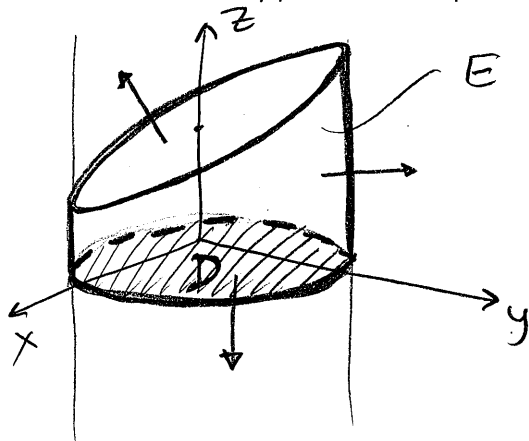
$$= \int_1^0 (1-u)u^2 - \frac{u^2}{2} + \frac{u^3}{3} du$$

$$= \int_1^0 \frac{u^2}{2} - \frac{2}{3}u^3 du$$

$$= \frac{u^3}{6} - \frac{u^4}{6} \Big|_1^0$$

$$= \boxed{0}$$

2. Verify your answer to problem 2 by using the Divergence Theorem (10 points)



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV = \iiint_E 3 \, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{9-3r\cos\theta} 3r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 3r(9-3r\cos\theta) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 27r \, dr \, d\theta \quad \text{--- 27 times area of D, or}$$

$$= \int_0^{2\pi} \left. \frac{27r^2}{2} \right|_0^1 d\theta \quad \leftarrow$$

$$= 2\pi \cdot \frac{27}{2}$$

$$= \boxed{27\pi}$$

3. What is another name for the Divergence Theorem? (1 point)

Gauss' Theorem OR Ostrogradsky's Theorem

Me after test 2: Can you curve my grade?

Jhevon:

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