Name: Jhevon Smith

Note that both sides of each page may have printed material.

If you could read the directions before asking me a question

THAT'D BE GREAT

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems.
4. Show **ALL** your work to receive full credit, unless otherwise stated. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. You are NOT allowed to use calculators, notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones and watches should be out of sight! If I catch you with a cellphone or a watch, you will fail the exam. If I catch you with any other paper than this exam, you will fail the exam. No calculators.**
9. Crossed-out or erased work will be ignored, even if correct. Problems where several unequal or contradictory answers are highlighted will be considered incorrect.
10. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
11. Other than that, have fun and good luck!

PROFESSORS WHO WRITE IMPOSSIBLY HARD EXAMS

JHEVON
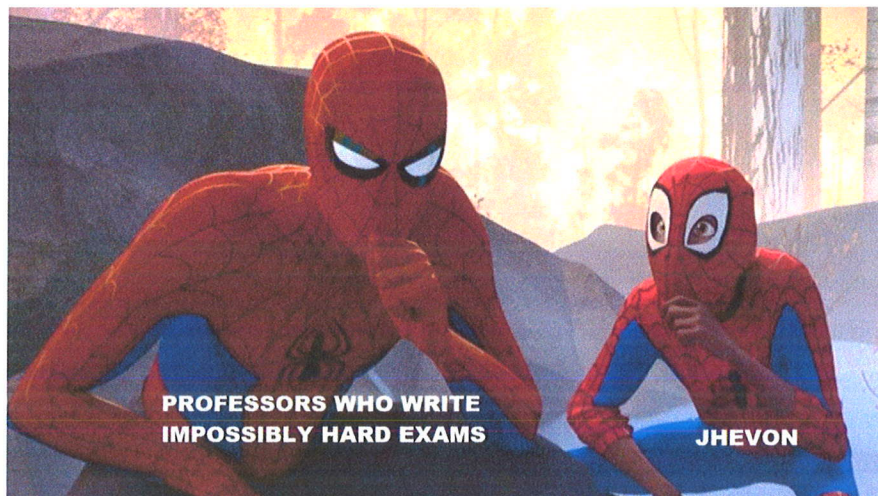
1. (15 points) Let S be the part of the plane $z = 1$ in the first octant bounded by $y = x^2$ and $y = 4$. Let C be the boundary curve of S. Compute
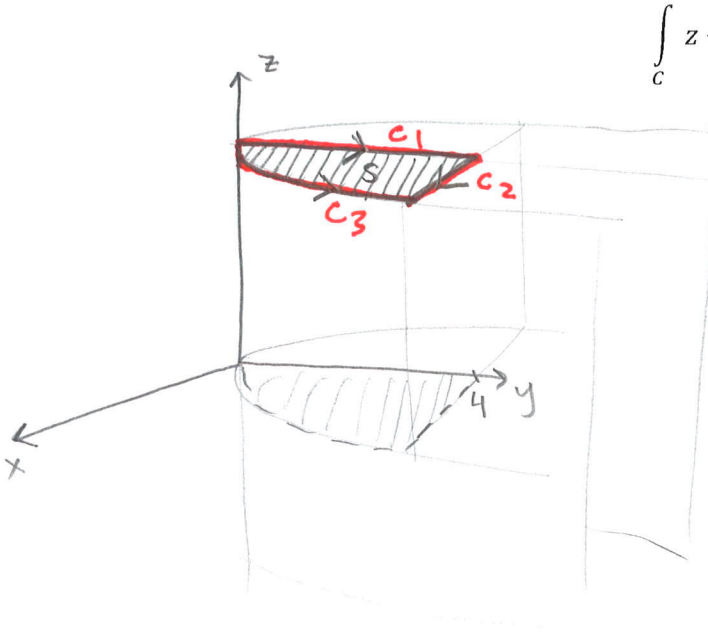
$$\int_C z + \sqrt{y} \, ds$$

(Orientation does not matter).

To be used later:

If $\tan\theta = 4$

$\Rightarrow$ (triangle with $\sqrt{17}$, 4, 1, angle θ)

$\Rightarrow \sec\theta = \sqrt{17}$

$\underline{C_1}:$ $\vec{r}(t) = \langle 0, 4t, 1 \rangle$, $0 \le t \le 1$

$\vec{r}'(t) = \langle 0, 4, 0 \rangle$

$\Rightarrow |\vec{r}'(t)| = 4$

$\Rightarrow \int_{C_1} = \int_0^1 (1 + \sqrt{4t})(4) \, dt$

$= \int_0^1 (4 + 8t^{1/2}) \, dt$

$= 4t + \frac{16}{3} t^{3/2} \Big|_0^1$

$= \boxed{4 + \frac{16}{3}}$ (5)

$\underline{C_2}:$ $\vec{r}(t) = \langle 2t, 4, 1 \rangle$, $0 \le t \le 1$

$\vec{r}'(t) = \langle 2, 0, 0 \rangle$

$\Rightarrow |\vec{r}'(t)| = 2$

$\Rightarrow \int_{C_2} = \int_0^1 (1 + \sqrt{4})(2) \, dt$

$= \int_0^1 6 \, dt$

$= \boxed{6}$ (5)

$\underline{C_3}:$ $\vec{r}(t) = \langle t, t^2, 1 \rangle$, $0 \le t \le 2$

$\vec{r}'(t) = \langle 1, 2t, 0 \rangle$

$\Rightarrow |\vec{r}'(t)| = \sqrt{4t^2 + 1}$

$\Rightarrow \int_{C_3} = \int_0^2 (1 + t)\sqrt{4t^2 + 1} \, dt$

$= \int_0^2 \sqrt{4t^2 + 1} \, dt + \int_0^2 t\sqrt{4t^2 + 1} \, dt$

For left integral: $2t = \tan\theta$
$\Rightarrow dt = \frac{1}{2}\sec^2\theta \, d\theta$
when $t = 2$, $\theta = \tan^{-1} 4$
when $t = 0$, $\theta = 0$

$\Rightarrow \int_0^{\tan^{-1}4} \frac{1}{2}\sec^3\theta \, d\theta$

$= \frac{1}{4}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) \Big|_0^{\tan^{-1}4}$

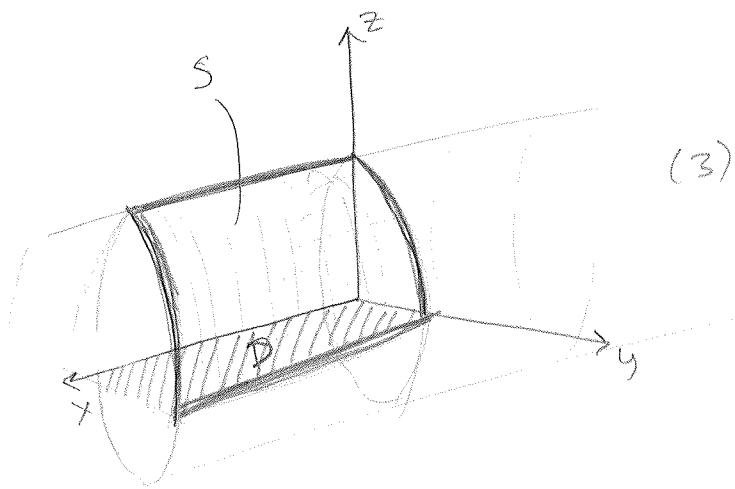$= \frac{1}{4}(4\sqrt{17} + \ln|\sqrt{17} + 4|) + \frac{\sqrt{17}^3}{12} - \frac{1}{12} = A$ (3)

For right integral: $u^2 = 4t^2 + 1$
$\Rightarrow 2u\,du = 8t \, dt$
$\Rightarrow \frac{1}{4} u \, du = t \, dt$
when $t = 2$, $u = \sqrt{17}$
when $t = 0$, $u = 1$

$\Rightarrow \int_1^{\sqrt{17}} \frac{1}{4} u^2 \, du$

$= \frac{1}{12} u^3 \Big|_1^{\sqrt{17}}$

$\Rightarrow \int_C = 4 + \frac{16}{3} + 6 + A$

$= \boxed{\frac{61}{4} + \frac{\sqrt{17}^3}{12} + \sqrt{17} + \frac{1}{4}\ln(4 + \sqrt{17})}$ (2)

2. (15 points) Compute $\iint\limits_{S} y^3 \, dS$, where S is the portion of $y^2 + z^2 = 1$ in the first octant, $0 \leq x \leq 4$.

Include a sketch of S in your answer.

(3)

$S: \; z = \sqrt{1-y^2}$

$\Rightarrow \vec{n} = \langle -f_x, -f_y, 1 \rangle$

$\Rightarrow \vec{n} = \langle 0, \frac{y}{\sqrt{1-y^2}}, 1 \rangle$

$\Rightarrow |\vec{n}| = \sqrt{\frac{y^2}{1-y^2} + 1}$

$= \sqrt{\frac{1}{1-y^2}}$

$= \frac{1}{\sqrt{1-y^2}}$ \qquad (3)

$\Rightarrow \iint\limits_{S} y^3 \, dS = \iint\limits_{D} y^3 \cdot \frac{1}{\sqrt{1-y^2}} \, dA$ \qquad (3)

$= \int_0^4 \int_0^1 \frac{y^3}{\sqrt{1-y^2}} \, dy \, dx$

(This is an improper integral, but in calculating, it will be clear it converges.)

$\rightarrow 4\int_0^1 \frac{y^2}{\sqrt{1-y^2}} \cdot y \, dy$

$u^2 = 1 - y^2 \Rightarrow y^2 = 1 - u^2$

$\Rightarrow 2u \, du = -2y \, dy$

$\Rightarrow -u \, du = y \, dy$

when $y = 1, \; u = 0$

when $y = 0, \; u = 1$

$= 4\int_1^0 \frac{1-u^2}{u} \cdot -u \, du$ \qquad (6)

$= 4\int_0^1 1 - u^2 \, du$

$= 4\left(u - \frac{u^3}{3}\right)\Big|_0^1$

$= 4\left(1 - \frac{1}{3}\right)$

$= \boxed{\frac{8}{3}}$

3. Let $F = \ <\cos(x+y) - x\sin(x+y)\ ,\ 2yz - x\sin(x+y)\ ,\ y^2\ >$.

(a) (6 points) Show that F is conservative by finding a general potential function, f, for F.

$\int x\sin(x+y)\,dx$

$\oplus$	1	$-\cos(x+y)$
$\ominus$	0	$-\sin(x+y)$

(2) $f = \int \cos(x+y) - x\sin(x+y)\,dx$

$= \sin(x+y) + x\cos(x+y) - \sin(x+y) + g(y,z)$

(1) also, $f = \int 2yz - x\sin(x+y)\,dy = y^2 z + x\cos(x+y) + h(x,z)$

(1) also, $f = \int y^2\,dz = y^2 z + j(x,y)$

(2) $\Rightarrow$ $\boxed{f = x\cos(x+y) + y^2 z + C}$

(b) (4 points) Find the work done by F in moving an object along the path C, where C is parametrized by two curves: the vector curve $\vec{r}(t) = \ <e^{t^2},\ t + 1,\ t^2\ >$ from $(1,1,0)$ to $(e, 2,1)$ (Note that $t = 0$ for the first point and $t = 1$ for the second point), followed by the straight line segment from $(e, 2,1)$ to $(0,1,1)$ (Remember how to parametrize line segments and how to set up the limits!!!).
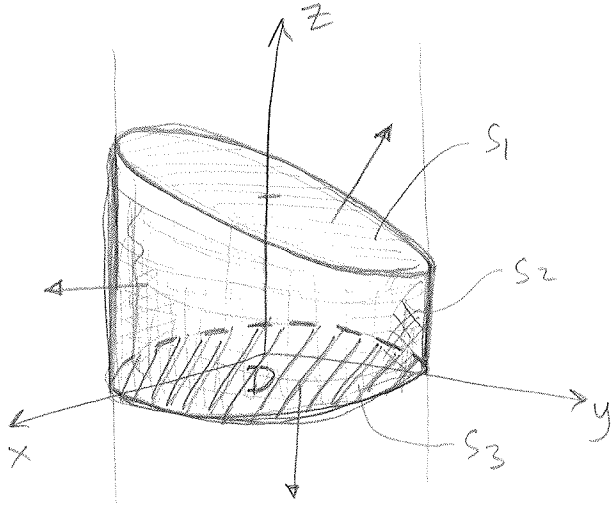
By the FTC for line Integrals:

$\int_C \vec{F} \cdot d\vec{r} = f(0,1,1) - f(1,1,0)$ (2)

$= 1 - (\cos 2)$

$= \boxed{1 - \cos 2}$ (2)

4. (20 points) Let E be the solid region bounded by the surfaces $y + 2z = 3$, $x^2 + y^2 = 9$, and the xy-plane. Let S be the positively oriented boundary surface of E. Let $\vec{F} = <x, y, z>$. Find the total flux through E using a surface integral.

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$\underline{S_1}: \vec{n} = <-f_x, -f_y, 1>, \quad f = \frac{3}{2} - \frac{y}{2}$

$= <0, \frac{1}{2}, 1>$

$\Rightarrow \iint_{S_1} = \iint_D \frac{y}{2} + z \, dA$

$= \iint_D \frac{y}{2} + \left(\frac{3}{2} - \frac{y}{2}\right) dA$

$= \iint_D \frac{3}{2} \, dA$

$= \frac{3}{2} \cdot \pi (3)^2$

$= \frac{27\pi}{2}$ \qquad (6)

$\underline{S_3}: \vec{n} = <0, 0, -1>$

$\Rightarrow \iint_{S_3} = \iint_D <x, y, z> \cdot <0, 0, -1> dA$

$= \iint_D (-z) \, dA$

$= \iint_D 0 \, dA$

$= 0$ \qquad (6)

$\underline{S_2}: \vec{r}(u,v) = <3\cos u, 3\sin u, v>$

$\qquad 0 \le u \le 2\pi, \quad 0 \le v \le \frac{3}{2} - \frac{3}{2}\sin u$

$\vec{r}_u = <-3\sin u, 3\cos u, 0>$

$\vec{r}_v = <0, 0, 1>$

$\rightarrow \vec{n} = <3\cos u, 3\sin u, 0>$

$\vec{F}(\vec{r}(u,v)) = <3\cos u, 3\sin u, v>$

$\Rightarrow \iint_{S_2} = \iint_R 9\cos^2 u + 9\sin^2 u \, dA$

$= \iint_R 9 \, dA$

$= \int_0^{2\pi} \int_0^{\frac{3}{2} - \frac{3}{2}\sin u} 9 \, dv \, du$

$= \int_0^{2\pi} 9\left(\frac{3}{2} - \frac{3}{2}\sin u\right) du$
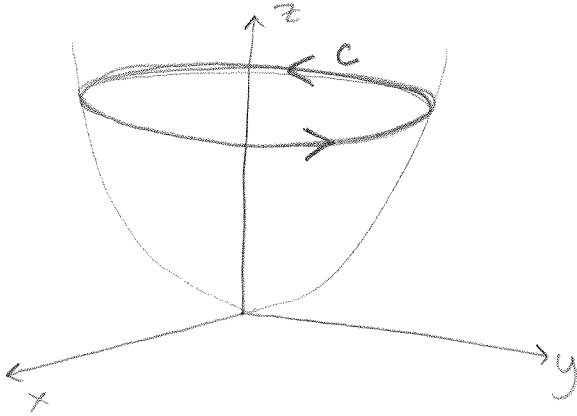
$= 2\pi \cdot \frac{27}{2}$

$= 27\pi$ \qquad (6)

$\Rightarrow \iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$

$= \frac{27}{2}\pi + 0 + 27\pi$

$= \boxed{\dfrac{81\pi}{2}}$ \qquad (2)

5. (20 points) Let $\vec{F} = y\,\mathbf{i} + x^2\,\mathbf{j} - z\,\mathbf{k}$. Let C be the curve of intersection of $z = x^2 + y^2$ and $z = 1$, oriented so that it is positively oriented when viewed from above. Using a line integral, find the work done by $\vec{F}$ in moving a particle once around C.

$C: \vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$ (3)

$\Rightarrow \vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$ (3)

$\Rightarrow \vec{F}(\vec{r}(t)) = \langle \sin t, \cos^2 t, -1 \rangle$ (3)

$\Rightarrow \text{Work} = \int_C \vec{F} \cdot d\vec{r}$

$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

$= \int_0^{2\pi} -\sin^2 t + \cos^3 t\, dt$ (3)

$= \int_0^{2\pi} -\tfrac{1}{2}(1 - \cos 2t) + (1 - \sin^2 t)\cos t\, dt$
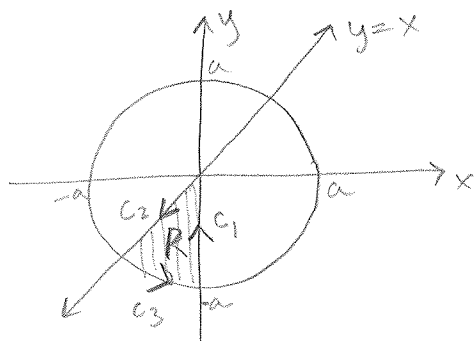
$= \int_0^{2\pi} -\tfrac{1}{2}\, dt$

$= \boxed{-\pi}$ (8)

6. Let R be the region in the third quadrant bounded by the circle of radius a, $y = x$, and the y-axis. Let C be the boundary curve of R, oriented *counter-clockwise*.

Compute $\oint_C y\,dx + x\,dy$ in two ways:

(a) Directly as a line integral. (10 points)

Intersection of $y = x$ w/ $x^2 + y^2 = a^2$

$\Rightarrow x^2 + x^2 = a^2$

$\Rightarrow x^2 = \left(\frac{a}{\sqrt{2}}\right)^2$

$\Rightarrow x = \pm\frac{a}{\sqrt{2}}$, we care about $x = -\frac{a}{\sqrt{2}}$

$\Rightarrow$ P.O.I. $\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$

$\underline{C_1}:\ x = 0,\quad y = -a + at,\ 0 \le t \le 1$

$dx = 0\,dt,\ dy = a\,dt$

$\Rightarrow \int_{C_1} = \int_0^1 0\,dt = 0 \quad (3)$

$\underline{C_2}:\ x = -\frac{a}{\sqrt{2}}t,\ y = -\frac{a}{\sqrt{2}}t,\ 0 \le t \le 1$

$dx = -\frac{a}{\sqrt{2}}dt,\ dy = -\frac{a}{\sqrt{2}}dt$

$\Rightarrow \int_{C_2} = \int_0^1 \frac{a^2}{2}t + \frac{a^2}{2}t\,dt$

$= a^2 \int_0^1 t\,dt$

$= \frac{a^2}{2} \quad (3)$

$\underline{C_3}\ x = a\cos t,\quad y = a\sin t,\ \frac{5\pi}{4} \le t \le \frac{3\pi}{2}$

$dx = -a\sin t\,dt,\ dy = a\cos t\,dt$

$\Rightarrow \int_{C_3} = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} -a^2\sin^2 t + a^2\cos^2 t\,dt$

$= a^2 \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \cos^2 t - \sin^2 t\,dt$

$= a^2 \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \cos 2t\,dt$

$= \frac{a^2}{2}\sin 2t \Big|_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}$

$= -\frac{a^2}{2} \quad (3)$

$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$
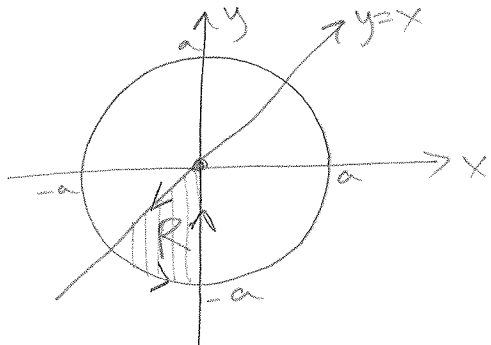
$= \frac{a^2}{2} + 0 - \frac{a^2}{2}$

$= \boxed{0} \quad (1)$

(This is problem 6 recopied for your convenience. Do part (b) on this page!)

Let R be the region in the third quadrant bounded by the circle of radius a, $y = x$, and the y-axis. Let C be the boundary curve of R, oriented *counter-clockwise*.

Compute $\displaystyle\oint_C y\,dx + x\,dy$ in two ways:

(b) As a double integral using Green's Theorem. (10 points)

By Green's Thm:

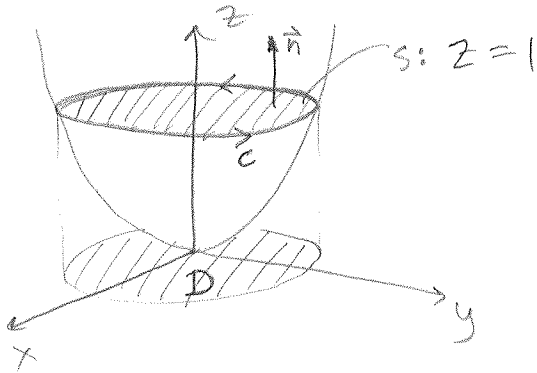$$\oint_C y\,dx + x\,dy = \iint_R Q_x - P_y \, dA$$

where $P = y$, $\quad Q = x$

$\Rightarrow P_y = 1$, $\quad Q_x = 1$

$$\Rightarrow \oint_C = \iint_R 0 \, dA$$

$$= \boxed{0}$$

1. Verify your answer to problem 5 by using Stokes' Theorem. (10 points)

By Stokes', $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}$

$$= \iint_D \text{curl} \vec{F} \cdot \vec{n} \, dA$$

$$\text{Now, curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 & -z \end{vmatrix}$$

$$= \langle 0, 0, 2x - 1 \rangle$$

and $\vec{n} = \langle 0, 0, 1 \rangle$

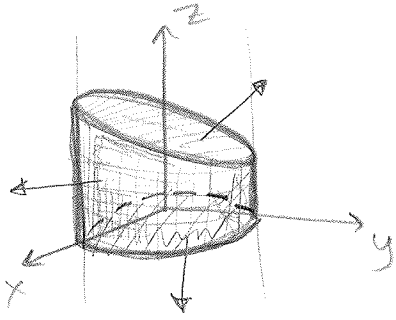$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \iint_D 2x - 1 \, dA$$

$$= \int_0^{2\pi} \int_0^1 (2r\cos\theta - 1) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (-1) dr \, d\theta$$

$$= \iint_D (-1) \, dA$$

$$= \boxed{-\pi}$$

2. Verify your answer to problem 4 by using the Divergence Theorem (10 points)

By the Divergence Theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} \, dV$$

$$= \iiint_E 3 \, dV$$

$$= \int_0^{2\pi} \int_0^3 \int_0^{\frac{3}{2} - \frac{1}{2} r \sin\theta} 3 r \, dz \, dr \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^3 \left(\frac{3}{2} - \frac{1}{2} r \sin\theta \right) r \, dr \, d\theta$$

$$= 3 \cdot 2\pi \int_0^3 \frac{3}{2} r \, dr$$

$$= 9\pi \cdot \left. \frac{r^2}{2} \right|_0^3$$

$$= \boxed{\frac{81\pi}{2}}$$

3. What is another name for the Divergence Theorem? (1 point)

Gauss' Theorem or Ostrogradsky's Theorem