Math 346 Test 3A
May 18, 2016

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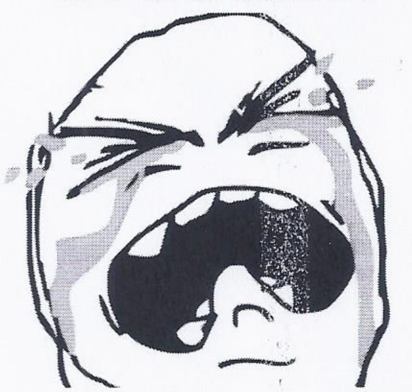
Note that both sides of each page may have printed material.

Instructions:

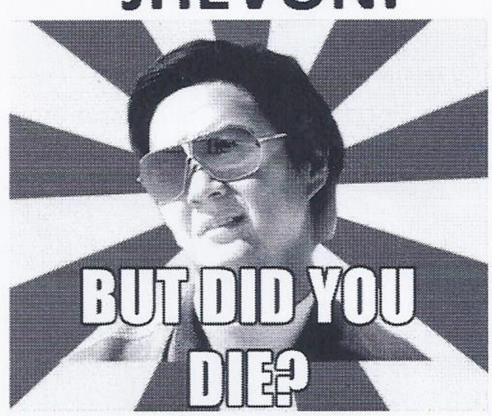
1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

STUDENT:



Jhevon chill! You make this class too hard. My GPA!!!

JHEVON:



1. Consider the matrix $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$.

(a) (10 points) Find the eigenvalues and corresponding eigenvectors of A.

$$\begin{vmatrix} \lambda-3 & -4 \\ 1 & \lambda+2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)(\lambda+2)+4=0$$
$$\Rightarrow \lambda^2-\lambda-2=0$$
$$\Rightarrow (\lambda-2)(\lambda+1)=0$$
$$\Rightarrow \boxed{\lambda_1=2,\ \lambda_2=-1} \rightarrow \text{Eigenvalues.}$$

$\lambda_1 = 2$

$$\left(\begin{array}{cc|c} 2-3 & -4 & 0 \\ 1 & 2+2 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_2 \\ R_1+R_2 \end{array}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4t \\ t \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} t$$

$\Rightarrow$ $\boxed{\begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_1 = 2.}$

$\lambda_2 = -1$

$$\left(\begin{array}{cc|c} -1-3 & -4 & 0 \\ 1 & -1+2 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_2 \\ \frac{R_1}{4}+R_2 \end{array}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$$

$\Rightarrow$ $\boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_2 = -1}$

(b) (10 points) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Set $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $P = (\vec{\lambda}_1 \mid \vec{\lambda}_2)$

$$\Rightarrow \boxed{D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},\ P = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}}$$

(Note: $P^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$, so we have $A = PDP^{-1}$
$= \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$)

(c) (10 points) Compute A^5.

By above $A = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$

$$\Rightarrow A^5 = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^5\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 32 \\ 1 & 4 \end{pmatrix} = \boxed{-\frac{1}{3}\begin{pmatrix} -129 & -132 \\ 33 & 36 \end{pmatrix}} = \boxed{\begin{pmatrix} 43 & 44 \\ -11 & -12 \end{pmatrix}}$$

2. (10 points) Using problem 1, solve the following system for the functions $y_1(t)$ and $y_2(t)$, subject to the initial conditions $y_1(0) = 1$ and $y_2(0) = 2$.

$$\begin{Bmatrix} y_1'(t) & = & 3y_1(t) + 4y_2(t) \\ y_2'(t) & = & -y_1(t) - 2y_2(t) \end{Bmatrix}$$

We have $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

From 1 we have $\lambda_1 = 2, \vec{\lambda}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\lambda_2 = -1, \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

$\Rightarrow$ $y_1 = -4c_1 e^{2t} - c_2 e^{-t}$

$y_2 = c_1 e^{2t} + c_2 e^{-t}$

$y_1(0) = 1 \Rightarrow 1 = -4c_1 - c_2$

$y_2(0) = 2 \Rightarrow 2 = c_1 + c_2$

$3 = -3c_1$

$c_1 = -1$

$\Rightarrow c_2 = 3$

$$\Rightarrow \boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4e^{2t} - 3e^{-t} \\ -e^{2t} + 3e^{-t} \end{pmatrix}}$$

3. Let $D: P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.

(a) (15 points) Find $[D]_B$, where $B = \{2, 2-3x, 2-3x+8x^2\}$ $= \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\8\end{pmatrix} \right\}$

$$[D]_B = \left[\left[D\begin{pmatrix}2\\0\\0\end{pmatrix}\right]_B \middle| \left[D\begin{pmatrix}2\\-3\\0\end{pmatrix}\right]_B \middle| \left[D\begin{pmatrix}2\\-3\\8\end{pmatrix}\right]_B \right]$$

$$= \left[\begin{pmatrix}0\\0\\0\end{pmatrix}_B \middle| \begin{pmatrix}-3\\0\\0\end{pmatrix}_B \middle| \begin{pmatrix}-3\\16\\0\end{pmatrix}_B \right]$$

$$= \boxed{\begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}}$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}_B$: $\begin{pmatrix}0\\0\\0\end{pmatrix}_B = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\begin{pmatrix}-3\\0\\0\end{pmatrix}_B$: $\begin{pmatrix}-3\\0\\0\end{pmatrix} = c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\-3\\0\end{pmatrix} + c_3\begin{pmatrix}2\\-3\\8\end{pmatrix}$

$\Rightarrow \begin{pmatrix}-3\\0\\0\end{pmatrix}_B = \begin{pmatrix}-3/2\\0\\0\end{pmatrix}$

$\begin{pmatrix}-3\\16\\0\end{pmatrix}_B$: $\begin{pmatrix}-3\\16\\0\end{pmatrix} = c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\-3\\0\end{pmatrix} + c_3\begin{pmatrix}2\\-3\\8\end{pmatrix}$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 2 & -3\\ 0 & -3 & -3 & 16\\ 0 & 0 & 8 & 0\end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -3/2\\ 0 & 1 & 1 & -16/3\\ 0 & 0 & 1 & 0\end{array}\right) \begin{matrix} R_1/2\\ R_2/-3\\ R_3/8\end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 23/6\\ 0 & 1 & 0 & -16/3\\ 0 & 0 & 1 & 0\end{array}\right) \begin{matrix} R_1 - R_2\\ R_2 - R_3\\ \end{matrix}$$

$$\Rightarrow \begin{pmatrix}-3\\16\\0\end{pmatrix}_B = \begin{pmatrix}23/6\\-16/3\\0\end{pmatrix}$$

(b) (10 points) Use part (a) to compute $D(6-6x+24x^2)$.

$$\begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}6\\-6\\24\end{pmatrix}_B = \begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 6\\ 0 & -3 & -3 & -6\\ 0 & 0 & 8 & 24\end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 3\end{array}\right)$$

$$= \boxed{\begin{pmatrix}13\\-16\\0\end{pmatrix}} = 13\begin{pmatrix}2\\0\\0\end{pmatrix} + (-16)\begin{pmatrix}2\\-3\\0\end{pmatrix} = \boxed{\begin{pmatrix}-6\\48\\0\end{pmatrix}}$$

In base B.

In standard bases.

(c) (15 points) Compute $[D(6-6x+24x^2)]_B$

$$[D(6-6x+24x^2)]_B = [D]_B[6-6x+24x^2]_B$$

$$= \begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}13\\-16\\0\end{pmatrix}}$$

4. (a) (10 points) Prove that if $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations of vector spaces, then the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

Pf: Since T_1 and T_2 are linear, we have $T_{1,2}(\vec{v}_1 + \vec{v}_2) = T_{1,2}(\vec{v}_1) + T_1(\vec{v}_2)$

and

$T_{1,2}(k\vec{v}_1) = k\, T_{1,2}(\vec{v}_1)$.

Then: ① $T_2 \circ T_1(\vec{v}_1 + \vec{v}_2) = T_2\big(T_1(\vec{v}_1 + \vec{v}_2)\big)$ — by defn of "∘"

$= T_2\big(T_1(\vec{v}_1) + T_1(\vec{v}_2)\big)$ — by addition property of T_1

$= T_2(T_1(\vec{v}_1)) + T_2(T_1(\vec{v}_2))$ — by addition property of T_2

$= T_2 \circ T_1(\vec{v}_1) + T_2 \circ T_1(\vec{v}_2)$ — by defn. of "∘"

② $T_2 \circ T_1(k\vec{v}_1) = T_2\big(T_1(k\vec{v}_1)\big)$ — by defn. of "∘"

$= T_2\big(k\, T_1(\vec{v}_1)\big)$ — by scalar property of T_1

$= k\, T_2(T_1(\vec{v}_1))$ — by scalar property of T_2

$= k\, T_2 \circ T_1(\vec{v}_1)$ — by defn of "∘"

$\Rightarrow T_2 \circ T_1$ is linear. $\blacksquare$

(b) (10 points) Prove that the property of being an isomorphism is *transitive*. That is, prove that if U, V and W are vector spaces, and U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

Pf: Let $T_1 : U \to V$, $T_2 : V \to W$ be isomorphisms. We need to show that there is an isomorphism from $U \to W$.

Claim: $T_2 \circ T_1$ is such an isomorphism.

Pf of claim: Clearly $T_2 \circ T_1$ goes from $U \to W$, and by part (a) it is a linear transformation. To see that it is one to one and onto, we need only note that the composition of two one to one and onto functions is itself one to one and onto (this is proven in the bonus). □

Since $T_2 \circ T_1$ is an isomorphism from $U \to W$, we have that U is isomorphic to W. $\blacksquare$

Bonus Problems:

1. Recall that a function $f: A \to B$ is **one-to-one** if and only if $f(x) = f(y) \Rightarrow x = y$. Also f is called **onto** if and only if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Prove the following:

   (a) (10 points) Let $f: A \to B$ and $g: B \to C$ be one-to-one functions. Prove that $g \circ f$ is one-to-one.

Pf: Assume f and g are 1-1, then $f(x) = f(y) \Rightarrow x = y$, and $g(a) = g(b) \Rightarrow a = b$.

Suppose $g \circ f(x) = g \circ f(y)$

$\Rightarrow g(f(x)) = g(f(y))$

$\Rightarrow f(x) = f(y)$ since g is 1-1 (put $a = f(x), b = f(y)$).

$\Rightarrow x = y$ since f is 1-1.

$\Rightarrow g \circ f$ is 1-1. $\blacksquare$

   (b) (10 points) Let $f: A \to B$ and $g: B \to C$ be onto functions. Prove that $g \circ f$ is onto.

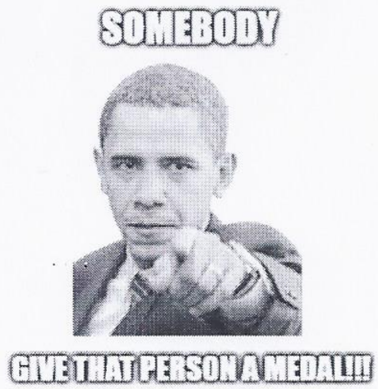
Pf: Assume f and g are onto. Then, for any $b_1 \in B$, there is $a \in A$ such that $f(a) = b_1$, and for any $c \in C$, there is $b_2 \in B$ such that $g(b_2) = c$.

Assume $c \in C$. Then there is $b \in B$, such that $g(b) = c$, since g is onto. But, since f is onto, there is $a \in A$ such that $f(a) = b$. That is, $g \circ f(a) = c$. Thus, for any $c \in C$, there is $a \in A$, such that $g \circ f(a) = c \Rightarrow g \circ f : A \to C$ is onto. $\blacksquare$

Note: to properly prove problem 4 part (b), you would have had to use the above two facts. If not, go back to 4(b) and try again.

You made it to the end of Math 346??

You've made your country proud!



Although…