Math 346 Test 2B

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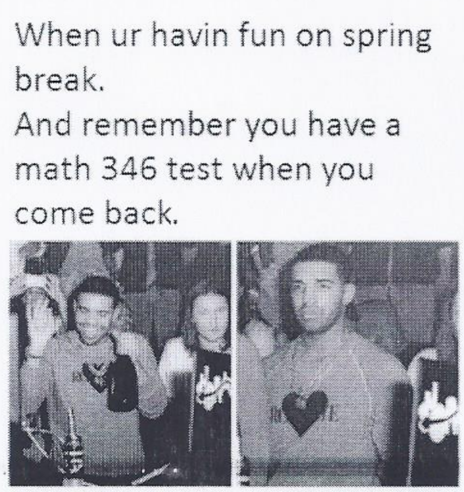
Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!



1. Consider bases $B = \{\vec{u}, \vec{v}\}$ and $B' = \{\vec{u}', \vec{v}'\}$ for $\mathbb{R}^2$ where $\vec{u} = \binom{2}{2}, \vec{v} = \binom{4}{-1}, \vec{u}' = \binom{1}{3}$, $\vec{v}' = \binom{-1}{-1}$.

(a) (5 points) Find the transition matrix $P_{B' \to B}$.

Using $\left(\begin{array}{c|c} \text{new basis} & \text{old basis} \end{array}\right)$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 0 & 5 & -2 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 10 & 0 & 13 & -5 \\ 0 & 1 & -2/5 & 0 \end{array}\right) \begin{array}{l} 5R_1 - 4R_2 \\ R_2/5 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 13/10 & -1/2 \\ 0 & 1 & -2/5 & 0 \end{array}\right) R_1/10$$

$$\Rightarrow \boxed{P_{B' \to B} = \begin{pmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{pmatrix}}$$

(b) (5 points) Find the transition matrix $P_{B \to B'}$.

$$\left(\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 0 & -2 & 4 & 13 \end{array}\right) \begin{array}{l} R_1 \\ 3R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & -5 \\ 0 & 1 & -2 & -13/2 \end{array}\right) \begin{array}{l} 2R_1 - R_2 \\ R_2/-2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & -5/2 \\ 0 & 1 & -2 & -13/2 \end{array}\right) R_1/2$$

$$\Rightarrow \boxed{P_{B \to B'} = \begin{pmatrix} 0 & -5/2 \\ -2 & -13/2 \end{pmatrix}}$$

↳ Note, you could also use $P_{B \to B'} = \left(P_{B' \to B}\right)^{-1}$

(c) (5 points) Compute the coordinate vector $[\vec{w}]_B$ where $\vec{w} = \binom{3}{-5}$.

Set $\binom{3}{-5} = c_1 \binom{2}{2} + c_2 \binom{4}{-1} = \begin{pmatrix} 2c_1 + 4c_2 \\ 2c_1 - c_2 \end{pmatrix}$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 4 & 3 \\ 2 & -1 & -5 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & 5 & 8 \end{array}\right) \begin{array}{l} R_1 \\ R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|c} 10 & 0 & -17 \\ 0 & 1 & 8/5 \end{array}\right) \begin{array}{l} 5R_1 - 4R_2 \\ R_2/5 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & -17/10 \\ 0 & 1 & 8/5 \end{array}\right) \begin{array}{l} R_1/10 \\ R_2 \end{array}$$

$$\Rightarrow \boxed{[\vec{w}]_B = \begin{pmatrix} -17/10 \\ 8/5 \end{pmatrix}}$$

(d) (5 points) Use either part (a) or part (b) to compute $[\vec{w}]_{B'}$

$$[\vec{w}]_{B'} = P_{B \to B'} [\vec{w}]_B = \begin{pmatrix} 0 & -5/2 \\ -2 & -13/2 \end{pmatrix} \begin{pmatrix} -17/10 \\ 8/5 \end{pmatrix} = \boxed{\begin{pmatrix} -4 \\ -7 \end{pmatrix}}$$

2. (a) (8 points) Prove that $W = \{A \in M_{22} : A^T = A\}$ is a subspace of M_{22}.

Pf: The vectors in W are of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

① Closure under addition: Let $\vec{x} = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix}, \vec{y} = \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \in W$.

$\Rightarrow \vec{x}+\vec{y} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{pmatrix}$. Since $(\vec{x}+\vec{y})^T = \vec{x}+\vec{y}$, $\vec{x}+\vec{y} \in W$.

② Closure under scalar multiplication: Let $\vec{x} = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \in W$.

$\Rightarrow k\vec{x} = \begin{pmatrix} ka_1 & kb_1 \\ kb_1 & kc_1 \end{pmatrix}$. Since $(k\vec{x})^T = k\vec{x}$, $k\vec{x} \in W$.

■

(b) (12 points) Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that $B = \{A_1, A_2, A_3\}$ is a basis for W defined above.

Pf: Set $c_1 A_1 + c_2 A_2 + c_3 A_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow c_1 = a,\ c_2 = c,\ c_3 = b$ — ①

$\Rightarrow \boxed{\{A_1, A_2, A_3\} \text{ spans } W.}$

Moreover, if we choose $a, b, c = 0$, then we have that $c_1 A_1 + c_2 A_2 + c_3 A_3 = \vec{0}$ has only the trivial sol'n, by equation ①.

$\Rightarrow \boxed{A_1, A_2, A_3 \text{ are linearly independent.}}$

■

3. Let $T: V \to W$ be a linear transformation.

(a) (6 points) Define $\ker T$ and prove that it is a subspace of V.

DEF: $\ker T = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$.

Pf: $\ker T$ is a subspace of V

① Closure under addition: If $\vec{x}, \vec{y} \in \ker T$, then

$$T(\vec{x}+\vec{y}) \overset{\text{Since } T \text{ is a lin. trans.}}{=} T(\vec{x}) + T(\vec{y}) = \vec{0} + \vec{0} = \vec{0}$$

$\Rightarrow \vec{x}+\vec{y} \in \ker T$.

② Closure under scalar multiplication:

If $\vec{x} \in \ker T$, then

$$T(k\vec{x}) = kT(\vec{x}) = k\vec{0} = \vec{0}$$

$\Rightarrow k\vec{x} \in \ker T$. ∎

(b) (6 points) Define $R(T)$ and prove that it is a subspace of W.

DEF: $R(T) = \{\vec{y} \in W \mid T(\vec{x}) = \vec{y} \text{ for some } \vec{x} \in V\}$.

Pf: $R(T)$ is a subspace of W

① Closure under addition: If $\vec{y}_1, \vec{y}_2 \in R(T)$, then there are $\vec{x}_1, \vec{x}_2 \in V$ such that $T(\vec{x}_1) = \vec{y}_1$ and $T(\vec{x}_2) = \vec{y}_2$. Since V is a vector space, $\vec{x}_1 + \vec{x}_2 \in V$ and $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2) = \vec{y}_1 + \vec{y}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 \in R(T)$.

② Closure under scalar multiplication: If $\vec{y} \in R(T)$, then there exists $\vec{x} \in V$ such that $T(\vec{x}) = \vec{y}$. Since V is a vector space, $k\vec{x} \in V$ and $T(k\vec{x}) = kT(\vec{x}) = k\vec{y} \Rightarrow k\vec{y} \in R(T)$. ∎

(c) (8 points) Suppose T is represented by a square matrix A. Prove that the following are equivalent:

(i) $\ker T = \{\vec{0}\}$

(ii) T is one-to-one

(iii) A is invertible

Pf: (i)$\Rightarrow$(ii): Assume $\ker T = \{\vec{0}\}$. Suppose $T(\vec{x}) = T(\vec{y})$. Then $T(\vec{x}) - T(\vec{y}) = T(\vec{x}-\vec{y}) = \vec{0}$. Since $\ker T = \{\vec{0}\}$, we must have $\vec{x}-\vec{y} = \vec{0} \Rightarrow \vec{x} = \vec{y} \Rightarrow T$ is 1-1.

(ii)$\Rightarrow$(iii): Assume T is 1-1, and $T(\vec{x}) = A\vec{x}$. If $T(\vec{x}) = \vec{y}$, then $A\vec{x} = \vec{y}$ and there is only one $\vec{x} \in \text{dom}(T)$ that maps to $\vec{y}$. $\Rightarrow A\vec{x} = \vec{y}$ has a unique soln. and so A is invertible.

(iii)$\Rightarrow$(i): Assume A is invertible. Then $A\vec{x} = \vec{0}$ has only the trivial soln. That is, if $T(\vec{x}) = \vec{0}$, then $\vec{x}$ must be $\vec{0} \Rightarrow \ker T = \{\vec{0}\}$. ∎

4. Let $A = \begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 2 & 4 & 7 & 7 & 0 \\ 3 & 6 & 9 & 9 & -1 \\ 1 & 2 & 4 & 4 & 1 \end{pmatrix}$

(a) (8 points) Find a basis for the column space of A

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 2 & 4 & 7 & 7 & 0 \\ 3 & 6 & 9 & 9 & -1 \\ 1 & 2 & 4 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \cdot -1 \\ R_4-R_2 \end{matrix} \Rightarrow \boxed{\text{Basis for col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_3-R_4$$

(b) (6 points) Find a basis for the null space of A

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = t,\ x_4 = r$

R_3: $x_5 = 0$

R_2: $x_3 = -x_4 = -r$

R_1: $x_1 = -2x_2 - 3x_3 - 3x_4$
$= -2t$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ -r \\ r \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\text{Basis for Null}(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}}$$

(c) (4 points) Find $rank(A)$ and $Nullity(A)$.

Rank(A) = # of pivots = 3

Nullity(A) = 2 → # of columns − rank, or dim(Null(A)),
or # of free variables in (b).

(d) (2 points) The correct answer to (c) is an example of a general principle. What is this principle?

The Dimension Theorem (for Matrices).

5. (a) (8 points) Let S be a finite set of linearly independent vectors. Prove that any non-empty subset of S is also a linearly independent set. Hint: a proof by contradiction is a good way to go.

Pf: Clearly the claim holds if $B = S$, where B is the subset in question. So assume $B \neq S$, $B \neq \emptyset$.

For convenience, write $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m\}$, $S = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m, \vec{b}_{m+1}, \dots, \vec{b}_n\}$

We show that B is a lin. indep. set. Assume to the contrary that B is dependent. Then there exists a non-zero soln to

$$c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_m\vec{b}_m = \vec{0} \text{ for the } c_i.$$

Using the same c_i's, note that the equation

$$c_1\vec{b}_1 + \dots + c_m\vec{b}_m + 0\cdot\vec{b}_{n+1} + \dots + 0\cdot\vec{b}_n = \vec{0}$$

has a non-trivial soln for the coefficients.

$\Rightarrow$ S is NOT linearly independent. $\lightning$ $\blacksquare$

(b) (6 points) Suppose $S = \{\vec{u}, \vec{v}\}$ is a linearly independent set. Prove that $B = \{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is also linearly independent.

Set $c_1(\vec{u}+\vec{v}) + c_2(\vec{u}-\vec{v}) = \vec{0}$.

$\Rightarrow (c_1+c_2)\vec{u} + (c_1-c_2)\vec{v} = \vec{0}$.

Since $\vec{u}, \vec{v}$ are lin. indep., the coefficients must be 0.

$\Rightarrow c_1+c_2=0$ and $c_1-c_2=0$. The soln to this system is $c_1=c_2=0$. So $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are lin. indep. $\blacksquare$

(c) (6 points) Let V be a vector space and $\vec{u}, \vec{v}, \vec{w} \in V$ be non-zero vectors. Can the set $\{\vec{v} - \vec{u}, \vec{w} - \vec{v}, \vec{u} - \vec{w}\}$ be linearly independent?

No! Notice that $1\cdot(\vec{u}-\vec{w}) + 1\cdot(\vec{v}-\vec{u}) + 1\cdot(\vec{w}-\vec{v}) = \vec{0}$.

So we have a nontrivial soln for the coefficients in the $\vec{0}$-linear combination. $\square$

(Alternatively, set $c_1(\vec{u}-\vec{w}) + c_2(\vec{v}-\vec{u}) + c_3(\vec{w}-\vec{v}) = \vec{0}$

$\Rightarrow (c_1-c_2)\vec{u} + (c_3-c_1)\vec{w} + (c_2-c_3)\vec{v} = \vec{0}$

and show there is a non-trivial soln to

$$c_1 - c_2 = 0$$
$$c_3 - c_1 = 0$$
$$c_2 - c_3 = 0 \text{).}$$

Bonus Problems (10 points each):

1. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation obtained by first reflecting over the x-axis, then rotating counter-clockwise by an angle of $\frac{\pi}{2}$, then projecting onto the y-axis. Find the standard matrix for T. Is T one-to-one?

Reflection over the x-axis $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Rotate by $\pi/2$ CCW $= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Project to y-axis $= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$

$\det[T] = 0 \Rightarrow \boxed{T \text{ is NOT } 1\text{-}1}$

2. Determine if the following linear operator T is one-to-one, and if possible, compute $T^{-1}(w_1, w_2, w_3)$. T is defined via

$$\begin{aligned} w_1 &= 2x_1 + 2x_2 + x_3 \\ w_2 &= 2x_1 + x_2 - x_3 \\ w_3 &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

Note: $[T] = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

Now $\det[T] = \det\begin{pmatrix} 2 & 2 & 1 \\ 4 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{vmatrix} 4 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

$\Rightarrow [T]$ is invertible $\Rightarrow \boxed{T \text{ is } 1\text{-}1}$

Now

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 1 & 3 & 0 & -2 \end{array}\right) \begin{array}{l} R_1 \\ R_1 - R_2 \\ 3R_1 - 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 3 & -1 & -2 & 2 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 \\ 2R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 3 & 0 & 5 & 1 & -4 \\ 0 & 0 & 1 & -1/3 & -2/3 & 2/3 \end{array}\right) \begin{array}{l} R_1/2 \\ 3R_2 - 2R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 5/3 & 1/3 & -4/3 \\ 0 & 0 & 1 & -1/3 & -2/3 & 2/3 \end{array}\right) \begin{array}{l} \\ R_2/3 \\ \\ \end{array}$$

$$\boxed{\Rightarrow [T]^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 5/3 & 1/3 & -4/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix}}$$

3. Suppose $L: P_2 \to P_2$ is given by $L(p(t)) = 2p(t) - 3p'(t)$. Given the basis $B = \{1, 3t+1, 2t^2\}$ for P_2,
(a) Find the B-coordinates of $p(t) = 5 - 6t + 4t^2$.
(b) Find $[L]_B$, the B-matrix for L.
(c) Use parts (a) and (b), compute $[L(5 - 6t + 4t^2)]_B$
Label what part you are doing!

Consider the isomorphism from $\mathbb{P}_2 \to \mathbb{R}_3$ by $a_0 + a_1 t + a_2 t^2 \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

(a) We want $\begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 3c_2 \\ 2c_3 \end{pmatrix}$

$\Rightarrow c_3 = 2, \ c_2 = -2, \ c_1 = 5 - c_2 = 7$

$$\Rightarrow \boxed{\begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}_B = \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}}$$

(b) $[L]_B = \left[\left[L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]_B \middle| \left[L\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}\right]_B \middle| \left[L\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\right]_B \right]$

$= \left[\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}_B \middle| \begin{pmatrix} -7 \\ 6 \\ 0 \end{pmatrix}_B \middle| \begin{pmatrix} 0 \\ -12 \\ 4 \end{pmatrix}_B \right]$

Note: If $p = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
then $L(p) = 2(1) - 3(0) = 2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
If $p = 3t+1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$
then $L(p) = 2(3t+1) - 3(3) = -7 + 6t = \begin{pmatrix} -7 \\ 6 \\ 0 \end{pmatrix}$
If $p = 2t^2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$
then $L(p) = 2(2t^2) - 3(4t) = -12t + 4t^2 = \begin{pmatrix} 0 \\ -12 \\ 4 \end{pmatrix}$

$= \left[\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} \right]$

① $2 = c_1 + c_2$, $0 = 3c_2$, $0 = 2c_3$ $\Rightarrow$ $c_1 = 2, c_2 = 0, c_3 = 0$

② $-7 = c_1 + c_2$, $6 = 3c_2$, $0 = c_3$ $\Rightarrow$ $c_1 = -9, c_2 = 2, c_3 = 0$

③ $0 = c_1 + c_2$, $-12 = 3c_2$, $4 = 2c_3$ $\Rightarrow$ $c_1 = 4, c_2 = -4, c_3 = 2$

$$\Rightarrow \boxed{[L]_B = \begin{pmatrix} 2 & -9 & 4 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}}$$

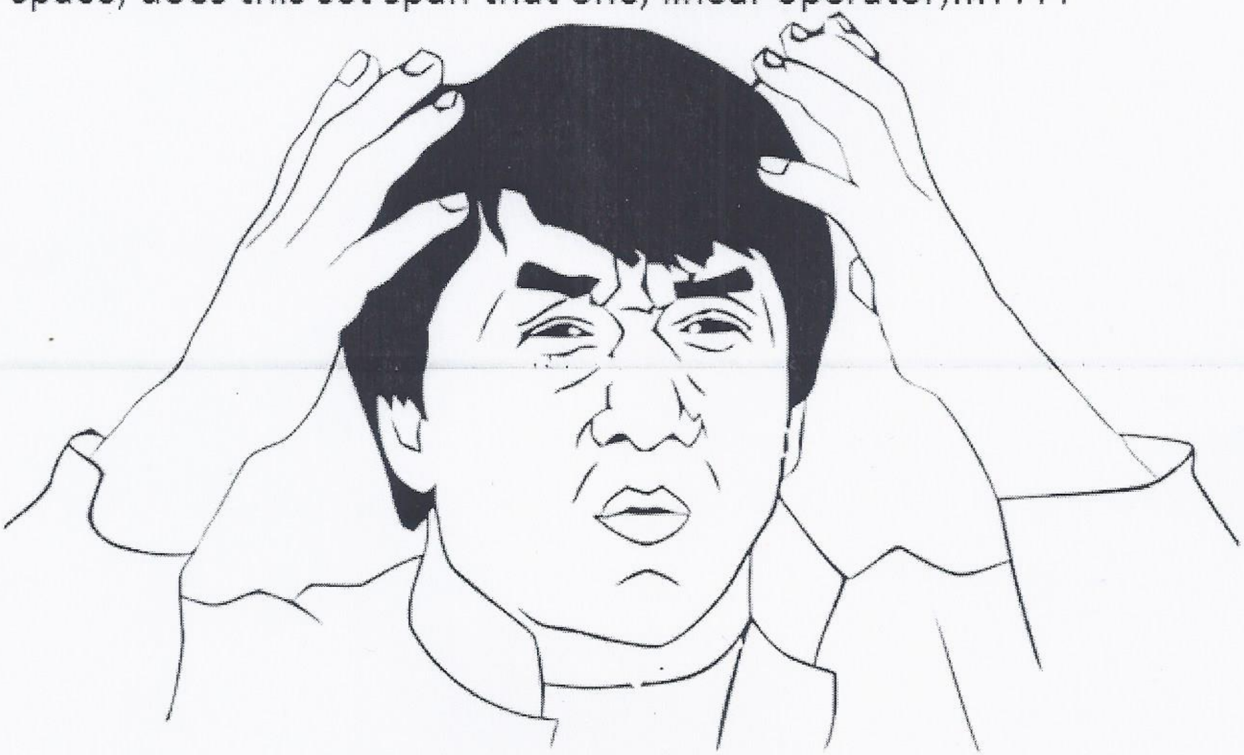
(c) $[L(5 - 6t + 4t^2)]_B = [L]_B [5 - 6t + 4t^2]_B$

$= \begin{pmatrix} 2 & -9 & 4 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 14 + 18 + 8 \\ 0 - 4 - 8 \\ 0 + 0 + 4 \end{pmatrix}$

$$= \boxed{\begin{pmatrix} 40 \\ -12 \\ 4 \end{pmatrix}}$$

?? Linear independence, span, basis, null space, subspace, vector space, does this set span that one, linear operator,...????



LINEAR ALGEBRA
Making people go crazy. Since 1973.
FML!