Math 346 Test 1A
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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.

2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!

3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.

5. Write neatly so that I am able to follow your sequence of steps and box your answers.

6. Read through the exam and complete the problems that are easy (for you) first!

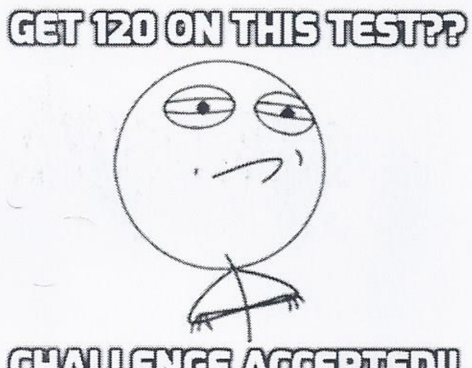
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.

8. In fact, **cell phones should be out of sight!**

9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

10. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.



1. Let $A = \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

(a) (15 points) Find A^{-1}

Using row reduction

$$\left(\begin{array}{ccc|ccc} -2 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} -R_3 \\ R_1+R_2 \\ 2R_3+R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}\right) \begin{array}{l} R_1-R_2 \\ -R_2 \\ R_2+R_3 \end{array}$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix}}$$

OR

Using the formula $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

$$\det A = \det \begin{pmatrix} 0 & -1 & 0 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} R_1+R_2$$

$$= 1 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$\operatorname{adj} A = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix}^T = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

matrix of cofactors

$$\Rightarrow A^{-1} = \frac{1}{1} \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix}}$$

Check: $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2-2+1 & 0+1-1 & +1-1+0 \\ 2-2+0 & 0+1+0 & 1-1+0 \\ -2+4-2 & 0-2+2 & -1+2+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓ check!

(b) (5 points) Use A^{-1} to solve the system $\begin{cases} -2x - z &= 1 \\ 2x - y + z &= -1 \\ -x + y &= 0 \end{cases}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1+1+0 \\ -1+1+0 \\ 1-2+0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}$$

2. Let $A = \begin{pmatrix} -2 & 3 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$

(a) (12 points) Find $\det A$

$$\det A = \det\begin{pmatrix} -2 & 3 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \quad R_3 + R_4$$

$$= -1 \cdot \begin{vmatrix} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= -1 \cdot 1 \cdot \begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix}$$

$$= \boxed{6}$$

(b) (2 points) Find $\det(A^{-2})$

$$\det(A^{-2}) = (\det A)^{-2} = 6^{-2} = \boxed{\frac{1}{36}}$$

(c) (2 points) Find $\det(A^{-1})$

$$\det(A^{-1}) = \frac{1}{\det A} = \boxed{\frac{1}{6}}$$

(d) (4 points) Find $\det(2A^3A^{-1}A^TA^{-2})$

$$\det(2A^3A^{-1}A^TA^{-2}) = 2^4(\det A)^3 \cdot \frac{1}{\det A} \cdot \det(A^T) \cdot (\det A)^{-2}$$

$$= \boxed{2^4 \cdot 6}$$

3. (a) (15 points) Solve the system by a method of your choice: $\begin{cases} 2x - 5y - 2z &= -1 \\ 7y + 4z &= 1 \\ x + y + z &= 0 \end{cases}$

Write your solution in vector form.

Using Gauss-Jordan Elimination / Row reduction

$$\left(\begin{array}{ccc|c} 2 & -5 & -2 & -1 \\ 0 & 7 & 4 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 7 & 4 & 1 \end{array}\right) \begin{array}{l} R_3 \\ R_2 \\ 2R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2/7 \\ R_2 - R_3 \end{array}$$

Set $z = t$

$R2 \Rightarrow y + \frac{4}{7}t = \frac{1}{7}$

$\Rightarrow y = \frac{1}{7} - \frac{4}{7}t$

$R1 \Rightarrow x + y + z = 0$

$\Rightarrow x = -y - z$

$= -\frac{1}{7} + \frac{4}{7}t - t$

$= -\frac{1}{7} - \frac{3}{7}t$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} - \frac{3}{7}t \\ \frac{1}{7} - \frac{4}{7}t \\ t \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -\frac{1}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{3}{7} \\ -4/7 \\ 1 \end{pmatrix}}$$

(b) (5 points) Could the above system be solved using Cramer's rule? Explain.

No. Since the RREF of the coefficient matrix is not I_n, the determinat is zero.

$\Rightarrow$ Cramer's rule cannot find the solution.

4. (a) Prove or disprove (2 points each):

(i) If A and B are invertible, then so is AB

This is true.

Pf: $(AB)^{-1} = B^{-1}A^{-1}$. We can see this since $(B^{-1}A^{-1})(AB) = B^{-1}I_nB$
$= B^{-1}B$
$= I_n$

$(AB)(B^{-1}A^{-1}) = I_n$ also. $\blacksquare$

(ii) If A and B are invertible, then so is $A - B$

This is not true.

Counter-example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then A^{-1} and B^{-1} exist, but $A - B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which is not invertible. $\square$

(iii) If $A^TA = A$, then A must be I_n

This is not true.

Counter-examples: If $A = (0)$ or $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $A^TA = A$ but $A \neq I_n$. $\square$

(iv) If A is invertible, then $\det(A^{-1}BA) = \det(B)$

This is true.

Pf: $\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A) = \frac{1}{\cancel{\det A}}\det B \cancel{\det A} = \det B$. $\blacksquare$

(v) If B is a square matrix, then BB^T is symmetric

This is true.

Pf: $(BB^T)^T = (B^T)^TB^T = BB^T$. $\blacksquare$

(vi) If B is a square matrix, then $B + B^T$ is symmetric

This is true.

Pf: $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$. $\blacksquare$

(vii) If B is a square matrix, then $B - B^T$ is skew symmetric

This is true.

Pf: $(B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T)$. $\blacksquare$

(b) True or false, circle "T" or "F" (2 points each):

(i) A linear system whose equations are all homogeneous is consistent. T F

(ii) Multiplying an equation in an augmented matrix by 0 is an elementary row operation. T F

(iii) A diagonal matrix with non-zero entries on the main diagonal is always invertible. T F

5. (a) (5 points) Let A, B and C be 3×3 matrices. Show that if all matrices are the same, except they differ in a single row, and that row of C can be obtained by adding the corresponding rows of A and B, then

$$\det C = \det A + \det B$$

(Hint: think of the cofactor expansion formula for a determinant)

Pf: Assume without loss of generality, that they differ in the first row. Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 & b_3 \\ x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix} \Rightarrow C = \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$

$$\Rightarrow \det C = (a_1+b_1)\begin{vmatrix} x_2 & x_3 \\ x_5 & x_6 \end{vmatrix} - (a_2+b_2)\begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} + (a_3+b_3)\begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix} \quad \text{(expand along } 1^{st} \text{ row).}$$

$$= \left(a_1\begin{vmatrix} x_2 & x_3 \\ x_5 & x_6 \end{vmatrix} - a_2\begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} + a_3\begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix}\right) + \left(b_1\begin{vmatrix} x_2 & x_3 \\ x_5 & x_6 \end{vmatrix} - b_2\begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} + b_3\begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix}\right)$$

$$= \det A + \det B.$$

$\blacksquare$

(b) (5 points) Use the above to show that, at least in the 3×3 case, a type 3 operation does not change the value of a determinant. You may assume, without proof, that the determinant of a matrix with two proportional rows is 0 (zero).

Pf: Assume without loss of generality (WLOG) that the type 3 operation performed on the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is adding a multiple of the second row to the first row.

$$\text{Then } \det\begin{pmatrix} a_{11}+ra_{21} & a_{12}+ra_{22} & a_{13}+ra_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \underbrace{= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \underbrace{\det\begin{pmatrix} ra_{21} & ra_{22} & ra_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{0 \text{ by assumption.}}}_{\text{by part (a)}}$$

$$= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\blacksquare$

6. (10 points) Consider the system in problem 1(b), that is, the system $\begin{cases} -2x - z &= 1 \\ 2x - y + z &= -1 \\ -x + y &= 0 \end{cases}$

Use Cramer's Rule to solve for x only. (Do NOT solve for y or z!) No credit will be given for any other method.

$$D = \begin{vmatrix} -2 & 0 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \boxed{0 & -1 & 0} \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} \; R_1 + R_2 \; = 1 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$D_x = \begin{vmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x = \frac{D_x}{D} = \frac{0}{1}$$

$$\Rightarrow \boxed{x = 0}$$

Bonus Problems:

Definition: Vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are said to be linearly independent if the only solution to the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

is the trivial solution, $c_1 = c_2 = \cdots = c_n = 0$.

Definition: A set B of vectors form a basis for a vector space if the set of vectors is (1) linearly independent and (2) they span the vector space—that is, every vector in the vector space can be expressed as a linear combination of vectors in B.

1. (5 points) Show that the functions x and x^2 are linearly independent.

Set $c_1 x + c_2 x^2 = 0 = 0 + 0x + 0x^2$

$\Rightarrow c_1 = c_2 = 0$ by equating coefficients.

$\Rightarrow x$ and x^2 are linearly independent. ■

2. (5 points) Show that the set $B = \{\vec{i}, \vec{j}\} = \{<1,0>, <0,1>\}$ is a basis for $\mathbb{R}^2$

Pf: ① linear independence

Set $c_1\vec{i} + c_2\vec{j} = \vec{0}$

$\Rightarrow c_1<1,0> + c_2<0,1> = <0,0>$

$\Rightarrow c_1 = c_2 = 0 \Rightarrow \vec{i}, \vec{j}$ are linearly independent.

② Span

Let $<x,y> \in \mathbb{R}^2$.

$\Rightarrow <x,y> = <x,0> + <0,y>$

$= x<1,0> + y<0,1>$

$= x\vec{i} + y\vec{j}$

$\Rightarrow \{\vec{i}, \vec{j}\}$ is a basis for $\mathbb{R}^2$. ■

3. (5 points) Prove that, in a vector space, $0\vec{u} = \vec{0}$.

Pf: $0\vec{u} = (0+0)\vec{u}$ by defn of 0.

$= 0\vec{u} + 0\vec{u}$ by distributive law

$\Rightarrow 0\vec{u} = \vec{0}$ by defn of $\vec{0}$. ■

4. (5 points) Prove that, in a vector space, $(-1)\vec{u} = -\vec{u}$

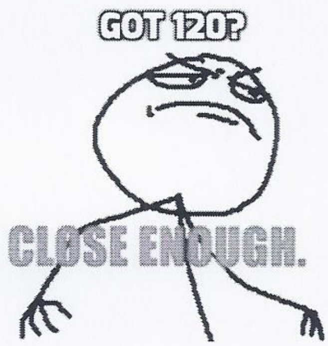
Pf: By the above, $\vec{0} = 0\vec{u}$

$= (1 + (-1))\vec{u}$

$= 1\vec{u} + (-1)\vec{u}$ by distributive law

$= \vec{u} + (-1)\vec{u}$ by $1\vec{u} = \vec{u}$ axiom.

$\Rightarrow (-1)\vec{u} = -\vec{u}$ by defn of $-\vec{u}$. ■


GOT 120?
CLOSE ENOUGH.