Math 308 Test 2

July 17, 2015

Name: ____

Instructions:

1. Read the instructions.

2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!

3. Complete all problems. In this exam. The weight of the problems are indicated.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)

5. Write neatly so that I am able to follow your sequence of steps. Be sure to explain yourself fully and clearly, using proper language. Any of your classmates should be able to read and understand your steps (in principle, it's best not to share answers :p)

6. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

7. Bring your fully solved tests with you when you come for the final on Monday. It is preferred that you somehow print this and write your solutions on it.

8. Other than that, have fun and good luck!

Remember: The force will be with you…always.

1.

(a) Prove that $\mathbb{R}$ is uncountable. (5 points)

(b) Prove that $\left|\mathbb{N}^{\mathbb{N}}\right| = |\mathbb{R}|$. (5 points)

2. Suppose A is a non-empty set of real numbers that has a lower bound. Prove that A has a greatest lower bound by using the completeness property of $\mathbb{R}$. (20 points)

3. (a) Let A and B be non-empty sets such that $|A| = |B|$. Prove that, if C is a non-empty set, then $|C^A| = |C^B|$. (15 points)

(b) Suppose now that $|A| \leq |B|$, prove that $|C^A| \leq |C^B|$. (5 points)

4. Prove that if $f: A \to B$ is surjective, then there exists $g: B \to A$ that is injective and such that $g \circ f = id_A$. (20 points)

5. Let A be a set of real numbers. We say that A is *dense* (in $\mathbb{R}$) iff for any two $x, y \in \mathbb{R}$ with $x < y$ there exists an $a \in A$ such that $x < a < y$. Assume A is countable. Prove that $R \setminus A$ is dense. (Hint: Use contradiction. Assuming the contrary implies that $(x, y) \subseteq A$—explain why this is! Of course, if $(x, y) \subseteq A$, then A cannot be countable—why? Therefore, $\mathbb{R} \setminus A$ must be dense.) (20 points)

6. (a) Prove that $\mathbb{Q}$ is countable. From scratch! (5 points)

(b) Prove that for any set A, $|A| < |P(A)|$. (5 points)

Bonus Problems:

1. (10 points) (a) Let $s: \mathbb{N} \to \mathbb{R}$ be a non-decreasing sequence, meaning $s_1 \leq s_2 \leq s_3 \leq \cdots$ Suppose that the sequence is bounded above, that is, there exists a real number M so that $s_n \leq M$ for all $n \in \mathbb{N}$. Prove that the sequence s_n is convergent. (What is the definition of a sequence converging to some real number?) Hint: $\{s_1, s_2, s_3, \dots\}$ is a set of real numbers. Say it's bounded above, by L. Show that $s_n \to L$.

(b) (5 points) The above result is actually a well-known theorem. What's the name of the theorem?

2. (15 points) Prove that, given any real number x, there exists a sequence $s\colon \mathbb{N} \to \mathbb{Q}$ (that is, every s_n is a rational number) and $s_n \to x$. For example, say $x = \pi$, then the sequence
$$s = \{(1,3), (2,3.1), (3,3.14), (4,3.141), (5,3.1415), \dots\} \text{ approaches } \pi.$$
Hint: Use the denseness of $\mathbb{Q}$. Notice that there must exist rational numbers $s_1, s_2, s_3, \dots,\ s_n, \dots$ in the intervals $(x-1, x+1)$, $\left(x-\frac{1}{2}, x+\frac{1}{2}\right)$, $\left(x-\frac{1}{3}, x+\frac{1}{3}\right)$, …, $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$,… Show that such a sequence approaches x.

Bonus points will first be applied to this test, if any are in excess, I will apply them to your first test. So, if you get 100 points for the non-bonus problems, and all 30 points for the bonus problems, 30 points will go to your first test grade.