Math 308 - Review for Test #1

`Problem 1a:` Explicitly write out the set $\{n \in \mathbb{Z} \mid n^2 < 5\}$.

`Problem 1b:` Determine the number of elements (called *cardinality*) of the set $\{0, 2, 4, ..., 2014\}$.

`Problem 1c:` Give an example of a set A, that has a subset B, such that $B \in A$.

`Problem 1d:` Let $S = \{0, 1, 2\}$, explicitly write out the set $\mathcal{P}(S)$ (the power set of S).

`Problem 1e:` Determine the number of elements (called *cardinality*) of the set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

`Problem 1f:` Let $A = \{\emptyset\}$. Write out explicitly $\mathcal{P}(\mathcal{P}(A))$.

`Problem 2:` Find the negations of the following statements, also determine if they are true/false:

- π is a rational number.
- For all $n \in \mathbb{N}, n + 1 \geq 2$.
- For every $x \in \mathbb{R}$, there is a $y > 0$, such that $xy = x$.
- For every $\varepsilon > 0$, there is a $\delta > 0$, such that $\delta < \varepsilon$.
- For any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that $\left|\sqrt[n]{2} - 1\right| < \varepsilon$, for all integers $n > N$.
- For any $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x| < \delta$, then $|\sin x| < \varepsilon$.

`Problem 3:` Recall that the *contrapositive* of the conditional statement $P \implies Q$ is the conditional statement $\neg Q \implies \neg P$. Construct a truth table for both of these compound statements and thereby show that they are logically equivalent.

`Problem 4:` Let $P(x) : x = -2$ and $Q(x) : x^2 = 4$ (where the variable x has domain $\mathbb{R}$).
Is $\forall x \in \mathbb{R}, P(x) \implies Q(x)$ true?
Is $\forall x \in \mathbb{R}, Q(x) \implies P(x)$ true?
Is $\forall x \in \mathbb{R}, P(x) \iff Q(x)$ true?

`Problem 5:` Let S be a finite set. The *cardinality* of S is the number of elements in S and it is denoted by $|S|$. Let A, B, C be finite sets. Prove that:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Hint: An idea: Let's say we wanted to prove $|A \cup B| = |A| + |B| - |A \cap B|$. You can prove this by noting that $|A \cup B| = |A| + |B - A|$ and $|A \cap B| + |B - A| = |B|$. These equations work since A and $B - A$ are disjoint, but their union is $A \cup B$; and $A \cap B$ and $B - A$ are disjoint, but their union is B. A similar technique could work here. Not the only way, but a nice way to do this.

`Problem 6a:` Prove that for all $n \in \mathbb{N}$, if $|n - 1| + |n + 1| \leq 1$ then $|n^2 - 1| \leq 4$.

`Problem 6b:` Prove that for all $x \in \mathbb{Z}$, if x is odd then $9x + 5$ is even.

`Problem 6c:` Prove that for all $n \in \mathbb{N}$, if $1 - n^2 > 0$ then $3n - 2$ is even.

`Problem 6d:` Prove that for all $x \in \mathbb{Z}$, $x^3 - x$ is even.

`Problem 7:` Let a be an integer, and n,m also integers. Suppose that $a|n$ and $a|m$ (this symbol means a divides n and a divides m). Prove that $a|(nx + my)$ for any two integers x, y.

`Problem 8:` Prove that if $a \equiv b(\text{mod} n)$ and $b \equiv c(\text{mod} n)$ then $a \equiv c(\text{mod} n)$.

`Problem 9:` Prove that for all real $a, b > 0$: $\frac{a}{b} + \frac{b}{a} \geq 2$.

`Problem 10:` Prove that for any $x \in \mathbb{R}$: $\sin^6 x + 3\sin^2 x \cos^2 x + \cos^6 x = 1$.
Hint: Consider $(\sin^2 x + \cos^2 x)^3$.

`Problem 11:` Use Venn diagrams to illustrate that $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$.
Then write a formal proof.

`Problem 12:` Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$.

`Problem 13:` Some topology. Use the notation (a, b) to denote the set $\{x \in \mathbb{R} \mid a < x < b\}$, called an *open interval.*

Let U be a subset of $\mathbb{R}$. We say that U is an *open* set, if for any $x \in U$ there is an open interval I such that $x \in I$ and $I \subseteq U$. A subset $C \subseteq \mathbb{R}$, is called *closed* if C is the complement of an open set i.e. $C = \mathbb{R} \setminus U$ for some open set U.

(i) Give example of a subset $A \subseteq \mathbb{R}$ which is neither open nor closed.
(i) Let U and V be open sets. Prove that $U \cap V$ is open.
(ii) Let C and D be closed sets. Prove that $C \cup D$ is closed.
(iii) Let $U_1, U_2, U_3, ...$ be a sequence of open sets. Prove that $\bigcup_{n=1}^{\infty} U_n$ is open.
(iv) Let $C_1, C_2, C_3, ...$ be a sequence of closed sets. Prove that $\bigcap_{n=1}^{\infty} C_n$ is closed.
(v) Give example of when $\bigcap_{n=1}^{\infty} U_n$ need not be open, and $\bigcup_{n=1}^{\infty} C_n$ need not be closed.

`Problem 14:` Prove by contradiction that there is no smallest positive rational number.

`Problem 15:` Prove that the product of an irrational number and a non-zero rational number is irrational. (How is the fact that the rational number being non-zero used in the proof?)

`Problem 16:` Prove that $\sqrt{3}$ is irrational by following a similar proof for $\sqrt{2}$ from class.
(You can use the fact: a^2 is divisible by 3 if and only if a is divisible by 3.)

`Problem 17:` Prove that the numbers $\log_2 3$ and $\log_3 2$ are irrational but their product is rational.

`Problem 18:` Show that there are no positive integers a, b such that $a^2 + 3 = 3^b$.

Hint: Think divisibility by 3.

`Problem 19a:` If a and b are integers such that $ab = 1$ prove that $a = b = 1$ or $a = b = -1$.

`Problem 19b:` Find the solutions to the equation $x^2 - 4y^2 = 1$ in terms of integers x and y.

`Problem 20a:` A polynomial $f(x)$ with integer coefficients is said to be *reducible* if and only if $f(x)$ factors $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are polynomials with integer coefficients with smaller degree than $f(x)$. For example, $x^2 - 1$ is reducible because $x^2 - 1 = (x + 1)(x - 1)$ and $x^3 + 1$ is reducible since $x^3 + 1 = (x + 1)(x^2 - x + 1)$. A polynomial $p(x)$ (with integer coefficients) is said to be *irreducible* iff it is not reducible. Prove that $x^2 + 1$ is irreducible by assuming it was possible to factor $x^2+1 = (x+a)(x+b)$, equating coefficients, and obtaining a contradiction.

`Problem 20b:` Prove that every polynomial $f(x)$ (with integer coefficients and degree $\deg f(x) > 1$) is a product of irreducible polynomials. (Hint: This is similar to the Fundamental Theorem of Arithmetic, every number $n > 1$ is a product of prime numbers. It is essentially the same argument).

`Problem 20c:` A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ is called *monic* iff its leading coefficient $a_n = 1$. So $x^2 + x + 1$ is monic while $2x^2 + x + 1$ is not. Prove that there are infinitely many irreducible monic polynomials. (Hint: Copy Euclid's proof that there are infinitely many primes by assuming there are only finitely many irreducible monic polynomials and obtaining a contradiction).

`Problem 21:` Let U be a set. For a subset X of U define its *complement*, written as X^C, as the set $X^C = U \setminus X$. Prove that if A and B are subsets of U then $(A \cap B)^C = A^C \cup B^C$. Prove this by writing out a formal proof and also by drawing a Venn diagram.

`Problem 22:` Prove that $x^{2015} + x^{2013} + x^{2011} + ... + x^5 + x^3 + x + 1 = 0$ has one real solution. Hint: You need to argue by contradiction and use Rolle's Theorem (or mean-value theorem).

`Problem 23:` Use mathematical induction to prove that $0^3 + 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2(n + 1)^2$.

`Problem 24:` Prove by induction that $1 + \dfrac{1}{\sqrt{2}} + ... + \dfrac{1}{\sqrt{n}} \geq \sqrt{n}$ for all integers $n \geq 1$.

`Problem 25:` Prove that $10|(3^{4n} - 1)$ for all $n \in \mathbb{N}$.

`Problem 26:` Prove that for all real numbers $a_1, a_2, ..., a_n$:

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

(The case when $n = 2$ we proved in class).

`Problem 27:` Define the *Fibonnaci sequence*: 1, 1, 2, 3, 5, 8, 13, 21,.... Where the pattern is that each succesive number is the sum of the two previous numbers. Mathematically if f_n denotes the n-th term (counting starts at 0 not 1) in the Fibonnaci sequence then $f_0 = f_1 = 1$ while $f_n + f_{n+1} = f_{n+2}$. Prove that f_n and f_{n+1} are *relatively prime*, meaning they share no common positive divisor other than 1. Hint: Suppose this statement was false, so there would be adjacent Fibonnaci numbers with a common positive divisor greater than 1. Define the set $S = \{n \geq 0 | f_n$ and f_{n+1} have a divisor greater than $1\}$. This set is non-empty and so by the well-ordering principle has a minimimal element m. Use this minimal element to derive a contradiction.

`Problem 28:` Prove that every positive integer is a sum of distinct Fibonnaci numbers.

`Problem 29:` Let a and b be natural numbers with $b > 0$. Prove that we can write $a = qb + r$ where $q \geq 0$ and $0 \leq r < b$. The numbers q is referred to as the *quotient* and r is referred to as the *remainder*. For example, if $a = 34$ and $b = 5$ then $34 = 6 \cdot 5 + 4$, so $q = 4$ and $r = 4$, note that $0 \leq r = 4 < b = 5$.

Hint: Let $S = \{a - nb : n \in \mathbb{N}$ and $a - nb \geq 0\}$. Show that S is a non-empty set of natural numbers. Therefore, by the well-ordering principle it has a least element, call it r. Argue by contradiction that $0 \leq r < b$. This means that $a - qb = r$ for some $q \in \mathbb{N}$.

`Problem 30:` Prove that $4^n > n^3$ for all $n \in \mathbb{N}$.

`Problem 31:` Recall that a set of real numbers is called *well-ordered* iff every (non-empty) subset has a least element. Prove that if A and B are well-ordered sets then $A \cup B$ is well-ordered.