**Math 308 - Review for Test #2 and the Final**

`Problem 1:` Let $A = \{1, 2, 3, 4\}$. Come up with examples of eight different relations: (1) A relation that is reflexive, symmetris cand transitive; (2) a relation that is not reflexive, not symmetric, not transitive; (3),(4),(5) relaitons that are exclusively one of reflexive, symmetric, transitive; (6),(7),(8) relations that posses two of the properties, but not the third.

`Problem 2:` Define a relation $R$ on $\mathbb{R}$ by $xRy$ iff $xy \geq 0$. Is $R$ symmetric? Transitive? Reflexive?

`Problem 3:` Define a relation $\sim$ on $\mathbb{Z}$ by defining $x \sim y$ iff $x^2 = y^2$. Prove that $\sim$ is an equivalence relation and describe the set $\mathbb{Z}/\sim$ of its equivalence classes. (Notation: Something like $\mathbb{Z}/n \equiv \mathbb{Z}_n$, it's just a way to write the set of equivalence classes obtained from an equivalence relation.)

`Problem 4:` Define a relation $\sim$ on $\mathbb{R}$ as follows, $\sim\ = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Q}\}$.
(i) Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.
(ii) Show that $[\sqrt{2}] \neq [\sqrt{3}]$.

`Problem 5:` Define a relation $\sim$ on $\mathbb{R}$ by $\sim\ = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z}\}$.
(i) Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.
(ii) Describe what every equivalence class looks like.
(iii) What is $\mathbb{R}/\sim$ topologically?

`Problem 6:` Let $A = (\mathbb{R} \times \mathbb{R}) \setminus \{(0, 0)\}$. Define a relation $\sim$ on $A$ by saying that, $(x_1, y_1) \sim (x_2, y_2)$ if and only if $(x_1, y_1) = (cx_2, cy_2)$ for some non-zero real number $c \in \mathbb{R}$.
(i) Prove that $\sim$ is an equivalence relation on $A$.
(ii) Given a point $(x, y) \in A$, describe the equivalence class $[(x, y)]$ geometrically.
(We often denote $A/\sim$ by $\mathbb{P}^1$ and call it the *real-projective line*. One can define the real-projective plane $\mathbb{P}^2$ in a similar way, and more generally $\mathbb{P}^n$. This is an important construction in geometry.)

`Problem 7:` Let $A = \mathbb{R} \times [-1, 1]$. Note, $[-1, 1] = \{y \in \mathbb{R} \mid -1 \leq y \leq 1\}$ and so $A = \{(x, y) \mid x \in \mathbb{R} \text{ and } -1 \leq y \leq 1\}$. Geometrically this looks like horizontal strip. Define the relation $\sim$ on $A$ by saying that $(a, b) \sim (c, d)$ if and only if $(a - c) \in \mathbb{Z}$ and $b = (-1)^{(a-c)}d$ (the sign is determined by the parity of $a - c$). You do not need to prove that $\sim$ is an equivalence relation on $A$ and can assume that it is in this problem. Given an element $(a, b) \in A$ describe its equivalence class $[(a, b)]$ geometrically.

`Problem 8a:` Show that,

$$f = \{\ (x, 2x + 1) \mid x \in \mathbb{R}\}$$

as a function $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

`Problem 8b:` Show that $g : \mathbb{Q} \setminus \{2\} \to \mathbb{Q} \setminus \{5\}$ defined by $g(x) = \frac{5x+1}{x-2}$ is a bijection.

`Problem 9:` Come up with an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective. Come up with an example that is injective but no surjective, and so forth, there will be four different examples.

`Problem 10:` Let $A$ and $B$ be sets . Denote $A^B$ as the set of all functions from $B$ into $A$. Explicitly write out the set $\{0,1\}^{\{a,b,c\}}$. (There will be $2^3 = 8$ such functions).

`Problem 11a:` Explicity write out a formula for $f^{-1}$ in Problem 8a.

`Problem 11b:` Explicity write out a formula for $f^{-1}$ in Problem 8b.

`Problem 12a:` Recall that a *permutation* on $X$, is a function $p : X \to X$ which is a bijection. We use the notation $S_n$ to denote the set of all permutations on $X = \{1, 2, ..., n\}$. Explicity write out $S_3$ as a set (it will have $3! = 6$ permutations).

`Problem 12b:` Find an example of two permutations, $p_1$ and $p_2$ in $S_3$, such that $p_1 \circ p_2 \neq p_2 \circ p_1$.

`Problem 13:` Let $f : A \to B$ and $g : A \to B$ be two functions. Prove that $f = g$ if and only if $f(a) = g(a)$ for any $a \in A$. (Keep in mind that $f$ and $g$ are sets, so to show that $f = g$ one is required to show that each is a subset of another.)

`Problem 14:` Suppose $f : A \to B$ is onto and $g : B \to C$ is onto, prove that $g \circ f : A \to C$ is onto.

`Problem 15:` Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Prove that $f$ and $g$ are bijective functions. Then prove that $f^{-1} = g$ and $g^{-1} = f$.

`Problem 16:` Let $f : A \to B$ be a function. Define a relation $\sim$ on $A$ as follows,

$$\sim_f \quad = \{(x, y) \in A \times A \mid f(x) = f(y)\}$$

(i) Show that $\sim_f$ is an equivalence relation on $A$.
(ii) Describe the equivalence class $[a]$ more explicitly.
(iii) Define the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $f(x, y) = xy$. Draw the picture of $(\mathbb{R} \times \mathbb{R})/\sim$ of the set of all equivalence classes. (Whatever picture you get it has to be a partition of the plane).
(iv) Given any set $A$ and any equivalent relation $\sim$ on it, show that there exists a set $B$ together with a function $f : A \to B$, such that $\sim \; = \; \sim_f$. (This may be difficult if you don't see what set to use. Hint: consider $B = A/\sim$ .)

`Problem 17:` Prove that the countable union of countable sets is countable. Hint: you can use a proof similar to the one that proves $|\mathbb{N}| = |\mathbb{Q}^+|$.

`Problem 18:` Prove that for any set $A$, $|A| \leq |A|$. Also prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$. Also, if $A \subseteq B$, then $|A| \leq |B|$.

`Problem 19:` Let $a > 0$. We will prove that $a^n = 1$ for all $n \in \mathbb{N}$ by strong induction. Clearly the equation is true at $n = 0$. Assume it is true for some $k$. Note that,

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}}$$

Assuming $a^k = 1$ and $a^{k-1} = 1$ we find that $a^{k+1} = \frac{1 \cdot 1}{1}$. What is the problem in this incorrect proof?

`Problem 20:` Prove that the set of irrational numbers is uncountable.
(The proof should be something simple. There is a theorem: If $A$ is a countable set then $|\mathbb{R} \setminus A| = |\mathbb{R}|$; but this theorem is complete overkill. Come up with a short proof that does not use such an advanced theorem.)

`Problem 21:` Recall that $2^{\mathbb{N}}$ denotes the set of all functions $f : \mathbb{N} \to \{0, 1\}$. Use Cantor's diagonal argument to prove that $2^{\mathbb{N}}$ is not countable. (We know that $2^{\mathbb{N}}$ has the same cardinality as $\mathbb{R}$ which is uncountable but here you are asked to use the diagonal argument.)

`Problem 22:` If $A$ and $B$ are uncountable sets, then is it true that $|A| = |B|$?

`Problem 23:` Prove that if $A$ and $B$ are uncountable then $A \times B$ is uncountable.

`Problem 24:` Use the Cantor-Schroeder-Bernstein Theorem to prove that $\left|\mathbb{N}^{\mathbb{N}}\right| = |\mathbb{R}|$.

`Problem 25:` Determine which infinite set is larger, $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{R}}$.

`Problem 26:` Let $\mathcal{C}$ be a collection of sets, use the notation $\bigcup \mathcal{C}$ to denote the union of all sets in $\mathcal{C}$. In Problem 17, you proved that if $\mathcal{C}$ is countable and every set in $\mathcal{C}$ is countable, then $\bigcup \mathcal{C}$ is countable. Give an example of $\mathcal{C}$, consisting of finite sets, such that $\bigcup \mathcal{C}$ is uncountable.

`Problem 27:` Let $f : \mathbb{R} \to \mathbb{R}$ be an *additive* function such that $f(x)$ is always a rational number. By *additive* we mean to say, $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$. Prove that there exists a real number $r \neq 0$, such that $f(r) = 0$. (Hint: First, show that $f(0) = 0$ and $f(-x) = -f(x)$. Second, assume by contradiction and use a cardinality argument.)

`Problem 28:` Let $a \in \mathbb{Z}^{\mathbb{N}}$, in other words, $a$ is an integer sequence. We say that $a$ is *eventually constant*, if there is an natural number $k$, so that $a_i = a_j$ for all $i, j \geq k$. Let $\mathcal{E}$ be the subset of $\mathbb{Z}^{\mathbb{N}}$ consisting of all eventually constant sequences. Prove that $\mathcal{E}$ is countable. (Hint: You can use the fact that $\mathbb{Z}^n$, the set of integer sequences of length $n$, is countable for every natural number $n$, without proof. Therefore, by taking the union of all $\mathbb{Z}^n$, and calling this set $F$, will be a countable too. Because a countable union of countable sets is countable. Every eventually constant sequence consists of a head and a tail. The head of the sequence is finite integer sequence, the tail is constant. This observation makes it possible to define an injection of $\mathcal{E}$ into a countable set.)

`Problem 29:` Let $f : [0, 1] \to [0, 1]$ be a continous function. Prove that there exists a number $a \in [0, 1]$ so that $f(a) = a$. (Hint: It has nothing to do with cardinality.)

`Problem 30:` Show that $\frac{\pi}{\sqrt{2}}$ and $\sqrt[3]{2} + \sqrt[4]{e}$ are irrational numbers. (Assume $e$ and $\pi$ are transcendental).

`Problem 31:` Determine whether or not the following number is irrational, $\cos\left(\frac{\pi}{9}\right)$.
(Hint: Triple Angle Identity)

`Problem 32:` Determine whether or not the following number is irrational,

$$\sqrt[3]{2 + \sqrt{5}} - \sqrt[3]{-2 + \sqrt{5}}$$

`Problem 33:` Let $A$ and $B$ be sets of real numbers, with $A \subseteq B$.
(i) Prove that $\sup(A) \leq \sup(B)$
(ii) Prove that $\inf(A) \geq \inf(B)$
(Note: **Do not** assume that the supremum nor infimum are real numbers, they could be $\pm\infty$).

`Problem 34:` Let $A$ be a set of real numbers with $A \neq \emptyset$. Prove that $\inf(A) \leq \sup(A)$.

`Problem 35:` Fix $y > 0$ to be a real number. Prove that given any real number $x$, there exists a real number $0 < \varepsilon < y$, such that $x + \varepsilon$ is a rational number.
(Hint: You need to use density of $\mathbb{Q}$)

`Problem 36:` Give an explicit example of a set of rational numbers which has a rational upper bound but no rational least upper bound. This shows that $\mathbb{Q}$ is not *complete* (which is why there is no theory of *rational analysis*, instead we have *real analysis*).

`Problem 37:` If $a$ and $b$ are algebraic numbers then, one can show that, $a + b$ is an algebraic number*. Furthermore, $a \cdot b$, and $\frac{a}{b}$ (with $b \neq 0$), are algebraic numbers also. Prove that if $a \neq 0$ is algebraic and $\alpha$ is transcendental then $a + \alpha$ and $a \cdot \alpha$ are transcendental numbers. (Hint: the proof is easier than you might think. Rather than consider polynomials, consider the results mentioned in this problem.)

`Problem 38:` Let $a > 0$ and $x > 0$ be real numbers. Prove that,

- If $x^2 < a$, then there is a rational number $q > 0$, so that $x^2 < q^2 < a$.
- If $x^2 > a$, then there is is a real number $x > z > 0$, so that $x^2 > z^2 > a$.

(Hint: For the first one. Try to find, $0 < \varepsilon < 1$, such that $(x + \varepsilon)^2 < a$. Use Problem 35, to show that we can arrange this $\varepsilon$ in such a way, such that $x + \varepsilon$ is rational. Now choose $q = x + \varepsilon$ and proof is complete. As an additional hint, use the inequality that $\varepsilon^2 < \varepsilon$, so $(x+\varepsilon)^2 < x^2 + (1+2x)\varepsilon$. Pick the $\varepsilon$ such that $x^2 + (1 + 2x)\varepsilon < a$. )

(Hint: For the second one. Try to find, $\delta > 0$, such that $(x - \delta)^2 > a$. To arrange this $\delta$ for the inequality $x^2 - 2x\delta + \delta^2 > a$, it is sufficient to note that, $x^2 - 2x\delta + \delta^2 > x^2 - 2x\delta$. Now try to arrange the $\delta$ so that $x^2 - 2x\delta > a$.)

`Problem 39:` Let $a > 0$ be a real number. Define the set $A = \{q \in \mathbb{Q} \mid q^2 < a\}$. Show that $A$ has a real upper bound and is non-empty. Let $x = \sup(A)$, this is a real number. Prove that $x^2 = a$, by using the previous problem. (This proves that square roots exist for positive reals, and that the square roots of positive reals are themselves positive. We've taken this fact for granted several times throughout the semester.)