**Math 308 Test 1**
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**Note that both sides of each page may have printed material.**

**Instructions:**

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems. In this exam, each non-bonus problem is worth 20 points. The weight of the bonus problems are indicated.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! $x^2$ and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

*Remember:* math is fun, math is beautiful, this test is *not* hard, there is no spoon.

1.

(a) Let $x, y \in \mathbb{Z}$. Prove that $(x+y)^2$ is even if and only if $x$ and $y$ are of the same parity.

Pf: ($\Leftarrow$) Assume $x, y$ are of the same parity. Then (i) $x, y$ are both even or (ii) $x, y$ are both odd.

(i) Let $x = 2k$, $y = 2l$ for $k, l \in \mathbb{Z}$. Then $(x+y)^2 = 2 \cdot 2(k+l)^2$ which is even.

(ii) Let $x = 2k+1$, $y = 2l+1$ for $k, l \in \mathbb{Z}$. Then $(x+y)^2 = 2 \cdot 2(k+l+1)^2$, which is even.

($\Rightarrow$) Assume $x$ and $y$ are of opposite parity. WLOG, let $x = 2k$, $y = 2l+1$ for $k, l \in \mathbb{Z}$. Then

$$(x+y)^2 = (2k+2l+1)^2 = 4k^2 + 4kl + 2k + 4kl + 4l^2 + 2l + 2k + 2l + 1$$
$$= 2(2k^2 + 2l^2 + 4kl + 2k + 2l) + 1$$

which is odd. ∎

(b) Let $x \in \mathbb{Z}$. Prove that $3x + 2$ is odd if and only if $5x + 11$ is even.

(Many ways to approach this!)

Pf: Assume $3x+2$ is odd, so that $3x+2 = 2k+1$ for $k \in \mathbb{Z}$.

Then,
$$3x+2 = 2k+1$$
$$\Leftrightarrow 5x+2 = 2k+2x+1$$
$$\Leftrightarrow 5x+11 = 2k+2x+10$$
$$\Leftrightarrow 5x+11 = 2(k+x+5).$$

Since $(k+x+5) \in \mathbb{Z}$, $5x+11$ is even.

∎

2. (a) Prove that $n! > 2^n$ for every integer $n \geq 4$.

Pf: We proceed by induction on $n$ for $P(n)$: "$n! > 2^n \ \forall n \in \mathbb{Z}, n \geq 4$". $P(4)$ asserts that $4! = 24 > 2^4 = 16$, which is true! Thus the base case is established.

Assume $P(n)$ holds for some $n \geq 4$. Then,

$$
\begin{aligned}
n! &> 2^n \\
\Rightarrow (n+1)! &> (n+1) \cdot 2^n \\
&= n \cdot 2^n + 2^n \\
&> 2^n + 2^n \quad (\text{since } n \geq 4 > 1) \\
&= 2^{n+1}
\end{aligned}
$$

Which establishes $P(n+1)$. ∎

(b) For each $n \in \mathbb{N}$, let $P(n)$: "$n^2 + 5n + 1$ is an even integer."

(i) Prove that $P(n) \Rightarrow P(n+1)$.

Assume $P(n)$ holds, so $n^2 + 5n + 1 = 2k$ for some $k \in \mathbb{Z}$.

$$
\begin{aligned}
\text{Then } (n+1)^2 + 5(n+1) + 1 &= n^2 + 2n + 1 + 5n + 5 + 1 \\
&= n^2 + 5n + 1 + 2n + 6 \\
&= 2k + 2n + 6 \\
&= 2(k + n + 3), \text{ which is even.}
\end{aligned}
$$

So that $P(n+1)$ holds. ∎

(ii) For which $n$ is $P(n)$ actually true?

None! In fact, $n^2 + 5n + 1$ is always odd! (One may check the cases for $n$ even or odd.)

(iii) What is the moral of this exercise?

The base case step is important in the induction principle/methodology!

3. Let $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$, then $3|ab$.

Hint: You may want to prove a lemma, that if $c \in \mathbb{Z}$, then $c^2 \equiv 0 \ (mod\ 3)$ or $c^2 \equiv 1 \ (mod\ 3)$. Two other results from the text may come in handy: (1) If $3|x$ or $3|y$, then $3|xy$, and (2) If 3 does not divide $x$, then $3|(x^2 - 1)$. These last two results were proven in the text. You may use them without proof.

We first prove the lemma. We have three cases, (i) $c = 3k$, (ii) $c = 3k+1$, (iii) $c = 3k+2$ for some $k \in \mathbb{Z}$:

In (i): $c = 3k \Rightarrow c^2 = 9k^2 = 3(3k^2) \Rightarrow c \equiv 0 \pmod 3$

In (ii): $c = 3k+1 \Rightarrow c^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 \Rightarrow c \equiv 1 \pmod 3$.

In (iii): $c = 3k+2 \Rightarrow c^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 \Rightarrow c \equiv 1 \pmod 3$.

Thus the lemma is established. $\square$

Pf: Assume $3 \nmid ab$. Then by result (1) $3 \nmid a$ and $3 \nmid b$.
Result (2) then gives that $3 | (a^2 - 1)$ and $3 | (b^2 - 1)$.

$\Rightarrow a^2 = 1 + 3n$ and $b^2 = 1 + 3m$ for $m, n \in \mathbb{Z}$.

$\Rightarrow a^2 + b^2 = 3(m+n) + 2$.

That is, $a^2 + b^2 \equiv 2 \pmod 3$.

But then the lemma yiedds that $a^2 + b^2$ cannot be the square of an integer, since $c^2 \equiv 2 \pmod 3 \Rightarrow c \notin \mathbb{Z}$. ∎

4. **Definition:** Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Then we define the *supremum* of $S$, denoted $supS$, to be the least upper bound of $S$. That is, (1) $supS \geq s, \forall s \in S$, and (2) if $x$ is another upper bound of $S$, then $supS \leq x$.

(a) Prove that if $supS \in S$, then $supS = maxS$. (Recall, $\max S \geq s\ \forall s \in S$, if it exists!)

Pf: First note that if $\sup S \in S$, then $\max S$ must exist. For assume not. Since $\sup S \in S$ and $\sup S \geq s\ \forall s \in S$, it fulfils the criteria of a maximum of $S$ $\lightning$.

So $\max S \in S$ and $\max S \geq s\ \forall s \in S$. In particular, $\max S \geq \sup S$. (This can also follow from the defn of supremum.)

But by part (1) of the defn of supremum, $\sup S \geq \max S$. Since $\max S \geq \sup S$ and $\max S \leq \sup S$, equality is established. $\blacksquare$

(b) Theorem: (Denseness of $\mathbb{Q}$) If $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Prove that for any distinct $a, b \in \mathbb{R}$, there are an infinite number of rational numbers strictly between $a$ and $b$.

Pf: Assume WLOG $a<b$. Then the denseness of $\mathbb{Q}$ property gives there is $r_1 \in \mathbb{Q}$ s.th $a < r_1 < b$. But then $r_1, b \in \mathbb{R}$ with $r_1 < b$. So the property again gives that there is $r_2 \in \mathbb{Q}$ s.th $r_1 < r_2 < b$. We proceed this way by induction. Assume we have found $r_1, \ldots, r_n \in \mathbb{Q}$ such that $a < r_1 < r_2 < \cdots < r_n < b$. Then the denseness of $\mathbb{Q}$ property shows $\exists\, r_{n+1} \in \mathbb{Q}$ such that

$a < r_1 < r_2 < \cdots < r_n < r_{n+1} < b$. Thus we may always find another rational between $a$ and $b$ when any finite number of them are found $\Rightarrow$ there are an infinite number of such rationals. $\blacksquare$

5. (a) A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for $a_n$ and prove that it holds using strong induction.

$a_1 = 1,\ a_2 = 2,\ a_3 = 2 + 2\cdot 1 = 4,\ a_4 = 4 + 2\cdot 2 = 8,\ a_5 = 8 + 2\cdot 4 = 16$

Conjecture: $a_n = 2^{n-1}$ for $n \in \mathbb{N}$. Let $P(n)$ be this statement.

We proceed using strong induction.
By defn, $P(1)$ and $P(2)$ are true; and we've shown $P(3)$ holds.
Assume $P(i)$ holds for all $1 \leq i \leq k$ where $k \geq 3$.
Then $a_k = 2^{k-1}$

$\Rightarrow a_{k+1} = a_k + 2a_{k-1}$

Since $k \geq 3$, $(k-1) \in \mathbb{N}$, and $k-1 < k$, which means the induction hypothesis holds.

$$\Rightarrow a_{k+1} = 2^{k-1} + 2\cdot 2^{k-2}$$
$$= 2^{k-1} + 2^{k-1}$$
$$= 2\cdot 2^{k-1}$$
$$= 2^k$$

$\blacksquare$

(b) For sets $A$ and $B$, prove that $A = (A - B) \cup (A \cap B)$.

Pf: Assume $x \in (A-B) \cup (A \cap B)$. Then $x \in A$ and $x \notin B$ or $x \in A$ and $x \in B$. In either case, $x \in A$. $\Rightarrow (A-B) \cup (A \cap B) \subseteq A$.

Now assume $x \in A$. Then we have two cases, (i) $x \in B$ or (ii) $x \notin B$.

In (i), if $x \in B$, then $x \in A \cap B \Rightarrow x \in (A \cap B) \cup (A - B)$, which means $A \subseteq (A \cap B) \cup (A - B)$.

In (ii), if $x \notin B$, then $x \in A - B \Rightarrow x \in (A-B) \cup (A \cap B)$, which means $A \subseteq (A-B) \cup (A \cap B)$.

In either case, we have the desired result.

$\blacksquare$

**Bonus Problems:**

1. (5 points) Let $A$ be a set. Define a *relation on A*.

A relation on a set $A$ is a subset of $A \times A$.

2. (5 points) Let $R$ be a relation on a set $A$. Define what it means for $R$ to be an equivalence relation on $A$?

A relation $R$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

- reflexive, $(a,a) \in R \; \forall a \in A$. $\longrightarrow$ notation $aRa \; \forall a \in R$
- symmetric, $(a,b) \in R \Rightarrow (b,a) \in R \rightarrow aRb \Rightarrow bRa$
- transitive, $(a,b) \wedge (b,c) \in R \Rightarrow (a,c) \in R \rightarrow aRb$ and $bRc \Rightarrow aRc$.

3. (10 points) Define a relation $R$ on $\mathbb{Z}$ by $aRb$ iff $a \equiv b \; (mod\; 3)$. Show that $R$ is an equivalence relation and find its equivalence classes.

Pf: Since $3|0$, $a \equiv a \pmod 3$ and so $aRa$. This gives us reflexivity.

For symmetry, assume $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$. So $aRb \Rightarrow bRa$.

For transitivity, we have proven in class that $a \equiv b \pmod 3$ and $b \equiv c \pmod 3 \Rightarrow a \equiv c \pmod 3$.
That is, $aRb$ and $bRc \Rightarrow aRc$. $\square$

Equivalence classes:

$[0] = \{0, \pm 3, \pm 6, \pm 9, \dots\} = \{n \in \mathbb{Z} \mid n = 3k \text{ for } k \in \mathbb{Z}\}$

$[1] = \{\pm 1, \pm 4, \pm 7, \dots\} = \{n \in \mathbb{Z} \mid n = 3k+1 \text{ for } k \in \mathbb{Z}\}$

$[2] = \{\pm 2, \pm 5, \pm 8, \dots\} = \{n \in \mathbb{Z} \mid n = 3k+2 \text{ for } k \in \mathbb{Z}\}$.

Note $[0], [1], [2]$ are pairwise disjoint and $[0] \cup [1] \cup [2] = \mathbb{Z}$.