MATH 209 TEST 1B

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Name: JHEVON SMITH

Note that both sides of each sheet has printed material

Instructions:

1. Read the instructions.
2. Don't panic!
3. Complete all problems! Bonus problems will not be counted unless all problems in the actual test are completed.
4. Note that each problem in the test is worth 20 points. The point value of the bonus problems are indicated.
5. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answer.
6. Write neatly, so that I am able to follow your sequence of steps. Indicate your answers by boxing them or otherwise.
7. Read through the exam and kill all the easy problems (for you) first!
8. Scientific calculators are needed, but you are NOT allowed to use notes, phones, iPads, telepathy, divine inspiration, or other outside aids–including, but not limited to, the smart kid that may be sitting beside you, or the friend you might be thinking of texting.
9. In fact, cell phones should be out of sight.
10. Use correct notation and write what you mean! "x^2" and "$x2$" are NOT the same thing, for example. And I will grade accordingly.
11. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.
12. Other than that, have fun, and good luck! :)

Remember: The force will be with you...always.

1. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{e^{3y}(1+x^4)}{x^4}$ subject to the initial condition $y(1) = 0$. Solve the equation explicitly, and then find the exact value of $y(0.5)$ correct to 3 decimal places.

$$\frac{dy}{dx} = \frac{e^{3y}(1+x^4)}{x^4}$$

$$\Rightarrow \frac{1}{e^{3y}}\,dy = \frac{1+x^4}{x^4}\,dx$$

$$\Rightarrow \int e^{-3y}\,dy = \int \left(\frac{1}{x^4}+1\right)dx$$

$$\Rightarrow -\frac{1}{3}e^{-3y} = -\frac{1}{3}x^{-3} + x + C$$

$$\Rightarrow e^{-3y} = x^{-3} - 3x + C$$

$$\Rightarrow -3y = \ln(x^{-3} - 3x + C)$$

$$\Rightarrow y = -\frac{1}{3}\ln(x^{-3} - 3x + C)$$

Since $y(1) = 0$, we get

$$0 = -\frac{1}{3}\ln(C-2)$$

$$\Rightarrow C - 2 = 1 \Rightarrow C = 3$$

$$\Rightarrow y = -\frac{1}{3}\ln(x^{-3} - 3x + 3)$$

$$\Rightarrow y(0.5) = -\frac{1}{3}\ln\left((0.5)^{-3} - 3(0.5) + 3\right)$$

$$\boxed{\approx -0.750}$$

2. Verify that $y = e^{2x}\left(2x - 2 + \frac{1}{x}\right) + \frac{e^2}{x}$ is a solution to the differential equation

$\frac{dy}{dx} = \frac{4x^2e^{2x} - y}{x}$ with the initial condition $y(1) = 2e^2$

Assume $y = e^{2x}(2x - 2 + 1/x) + e^2/x$

$\Rightarrow y(1) = e^2(2-2+1) + e^2/1$

$= e^2 + e^2$

$= 2e^2$

$\Rightarrow$ Initial condition verified!

We now verify this satisfies the ODE.

$$\text{LHS} = \frac{dy}{dx}$$

$$= 2e^{2x}(2x-2+1/x) + e^{2x}(2 - x^{-2}) - e^2x^{-2}$$

$$= 4xe^{2x} - 4e^{2x} + \frac{2}{x}e^{2x} + 2e^{2x} - \frac{1}{x^2}e^{2x} - \frac{e^2}{x^2}$$

$$= \frac{1}{x^2}\left(4x^3e^{2x} - 2x^2e^{2x} + 2xe^{2x} - e^{2x} - e^2\right)$$

$$\text{RHS} = \frac{4x^2e^{2x} - \left[e^{2x}(2x-2+1/x) + e^2/x\right]}{x}$$

$$= \frac{4x^2e^{2x} - 2xe^{2x} + 2e^{2x} - \frac{1}{x}e^{2x} - e^2/x}{x} \cdot \frac{x}{x}$$

$$= \frac{4x^3e^{2x} - 2x^2e^{2x} + 2xe^{2x} - e^{2x} - e^2}{x^2}$$

$= \text{LHS}$

Since RHS = LHS, the solution is verified!

3. Use Euler's method with 4 steps to estimate the solution at $x = 2$ to the differential equation $\frac{dy}{dx} = \frac{6x^2}{y}$ subject to $y(1) = 1$. Make a table showing your calculations. Your estimate should use 3 decimal places.

$a=1$, $b=2$, $\Delta x = \frac{b-a}{n}$

$= \frac{2-1}{4}$

$= 0.25$

x	y	y'
1	1	$\frac{6(1)^2}{1} = 6$
1.25	$1+6(0.25) = 2.5$	$\frac{6(1.25)^2}{2.5} = 3.75$
1.5	$2.5+3.75(0.25) = 3.4375$	$\frac{6(1.5)^2}{3.4375} = 3.9273$
1.75	$3.4375+3.9273(0.25) = 4.4193$	$\frac{6(1.75)^2}{4.4193} = 4.1579$
2	$4.4193+4.1579(0.25) = 5.4588$	

$\Rightarrow \boxed{y(2) \approx 5.459 \text{ to 3 d.p.}}$

4. Consider the differential equation $\frac{dy}{dx} = \frac{6x^2}{y}$ with initial condition $y(1) = 1$. Use the modified Euler's method with step size 0.5 to approximate the value of $y(2)$ to 3 decimal places. Make a table to show your calculations. (Drawing the table landscape may be best.)

We are given $\Delta x = 0.5$

x	y	y'	$x_{1/2}$	$y_{1/2}$	$y'_{1/2}$
1	1	$\frac{6(1)^2}{1} = 6$	1.25	$1 + 6(0.25) = 2.5$	$\frac{6(1.25)^2}{2.5} = 3.75$
1.5	$1 + 3.75(0.5) = 2.875$	$\frac{6(1.5)^2}{2.875} = 4.6957$	1.75	$2.875 + 4.6957(0.25) = 4.0489$	$\frac{6(1.75)^2}{4.0489} = 4.5383$
2	$2.875 + 4.5383(0.5) = 5.1442$				

$\Rightarrow$ $\boxed{y(2) \approx 5.144 \text{ to 3 d.p.}}$

5. After carefully examining the lifespans in his family tree, John is convinced that he will die at age 89. He wishes to retire at age 65 and live (with wanton disregard) on \$70,000 per year. When he retires, John plans to put his entire settlement into a single investment account (wanton disregard, remember?) that is supposed to yield 3.5% interest per year compounded continuously. Once invested, he will start withdrawing his \$70,000 per year to do all the things on his bucket list.

(a) Suppose P represents the balance in the account at time t (in years). Obtain an equation for $\frac{dP}{dt}$.

$$\frac{dP}{dt} = 0.035P - 70000$$

(b) Solve the equation in part (a) with a yet undetermined initial value P_0.

$$\frac{dP}{dt} = 0.035(P - 2000000)$$

$$\Rightarrow \int \frac{dP}{P - 2000000} = \int 0.035\, dt$$

$$\Rightarrow \ln|P - 2000000| = 0.035t + C$$

$$\Rightarrow P = Ce^{0.035t} + 2000000$$

$$P(0) = P_0$$

$$\Rightarrow P_0 = Ce^{0} + 2000000$$

$$\Rightarrow C = P_0 - 2000000$$

$$\Rightarrow \boxed{P = (P_0 - 2000000)e^{0.035t} + 2000000}$$

(c) How large must John's retirement settlement be if he plans to withdraw his bucket-list money until he dies at the expected time?

We want $P(24) = 0$

$$\Rightarrow 0 = (P_0 - 2000000)e^{0.035(24)} + 2000000$$

$$\Rightarrow P_0 = \frac{-2000000}{e^{0.035(24)}} + 2000000$$

$$\approx 1136578.95$$

So he needs a settlement of $\boxed{\$1136578.95}$

Work hard, John!

Bonus 1: (5 points) Find the exact solution (to 3 decimal places) to problem 3.

$$\frac{dy}{dx} = \frac{6x^2}{y}$$

$$\Rightarrow \int y\,dy = \int 6x^2\,dx$$

$$\Rightarrow y^2/2 = 2x^3 + C$$

$$\Rightarrow y^2 = 4x^3 + C$$

$$\Rightarrow y = \pm\sqrt{4x^3 + C}$$

$$y(1) = 1 \Rightarrow 1 = \sqrt{4(1) + C}$$

$$\Rightarrow C = -3$$

$$\text{So } y = \sqrt{4x^3 - 3}$$

$$\Rightarrow y(2) = \sqrt{29} \approx \boxed{5.385} \text{ to 3 d.p.}$$

Bonus 2: (15 points) Consider the differential equation $\dfrac{dy}{dx} = y(y-4)$. Use qualitative analysis as done in class to draw the solution curves for $y(0) = -1$, $y(0) = 1$, $y(0) = 3$, and $y(0) = 5$. You don't need to show how you get concavity, but show the rest of your work, including a fully labeled stability diagram.

① Steady states and inflections

S.S: $y(y-4) = 0$

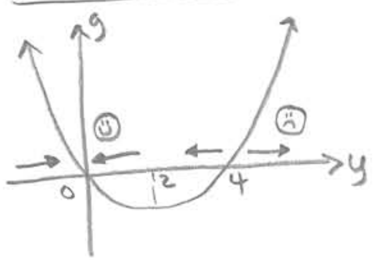
$y = 0,\ y = 4$

I.P: $g = y^2 - 4y$

$\Rightarrow g' = 2y - 4$

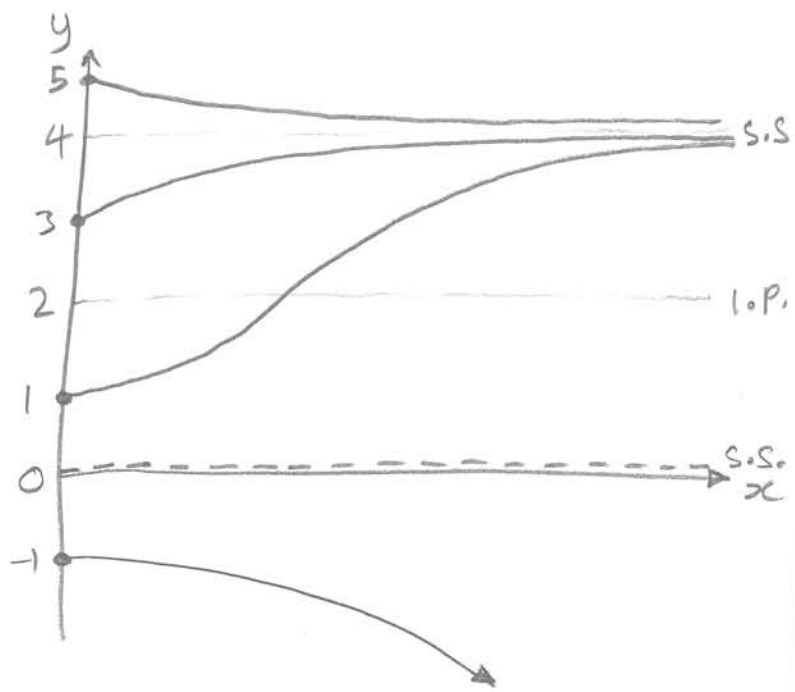
$\Rightarrow g' = 0 \Rightarrow y = 2$

② Stability graph



$\Rightarrow$

④ Solution graphs



③ Concavity

Skip!