Math 205 Test 3

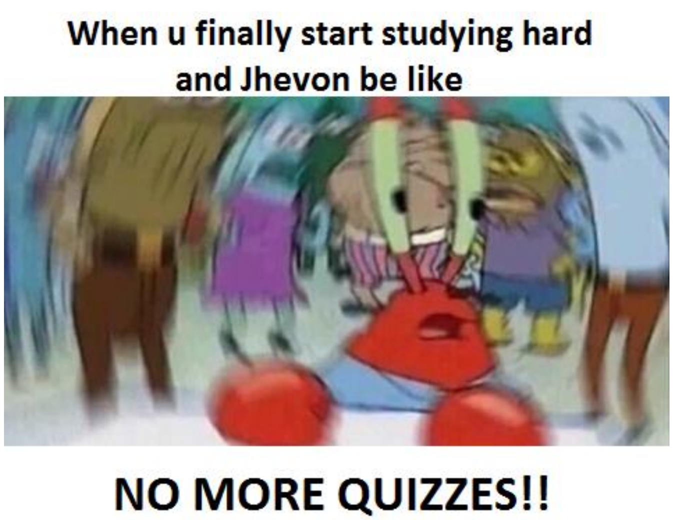
November 30, 2016

Name: ___________________________SOLUTIONS____________________________________

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!



1. (4 points each part) A population $P(t)$ doubles every 3 years. In 1990, the population was 4 million. Assume exponential growth and that we start measuring the population in 1990.

(a) Find the differential equation satisfied by $P(t)$. Also include the initial condition.

Since doubling time $= \frac{\ln 2}{r} \Rightarrow r = \frac{\ln 2}{3}$

$$\Rightarrow \boxed{P' = \frac{\ln 2}{3} P}, \quad \boxed{P(0) = 4 \text{ million}}$$

Differential eq.   Initial condition

(b) Find and simplify $P(t)$

$$\boxed{P = 4 \cdot e^{\frac{\ln 2}{3} t}} \quad \text{or} \quad \boxed{P = 4 \cdot 2^{t/3}}$$

In millions   In millions

(c) What is the population's size in 1992?

We want $P(2)$

$$\boxed{P(2) = 4 e^{\frac{\ln 2}{3}(2)}} \quad \text{or} \quad \boxed{P = 4 \cdot 2^{2/3}}$$

In millions   In millions

(d) After how long will the population be 10 million?

We want t when

$$10 = 4 e^{\frac{\ln 2}{3} t} \quad \text{or} \quad 10 = 4 \cdot 2^{t/3}$$

$$\Rightarrow \ln \frac{5}{2} = \ln e^{\frac{\ln 2}{3} t} \quad \text{or} \quad \Rightarrow \log_2 \frac{5}{2} = \log_2 2^{t/3}$$

$$\Rightarrow \ln \frac{5}{2} = \frac{\ln 2}{3} t \quad \text{or} \quad \Rightarrow \log_2 \frac{5}{2} = t/3$$

$$\Rightarrow \boxed{t = \frac{3 \ln \frac{5}{2}}{\ln 2}} \text{ years.} \quad \text{or} \quad \Rightarrow \boxed{t = 3 \log_2 \frac{5}{2}}$$

(e) How fast is the population growing when it reaches 6 million?

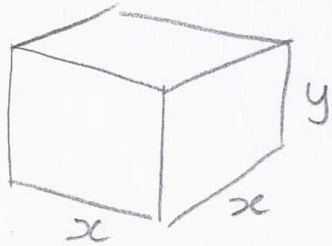
Since $P' = \frac{\ln 2}{3} P$

$$\Rightarrow \boxed{P' = \frac{\ln 2}{3}(6)} \text{ million/yr} \quad \text{or} \quad \boxed{P' = 2 \ln 2} \text{ million/yr}$$

2. (20 points) A closed rectangular box with a square base and volume 12 cubic feet is to be constructed using two different types of materials. The top is made of metal costing \$2 per square foot, and the remaining sides and base is made of wood costing \$1 per square foot. Find the dimensions of the box for which the cost of construction is minimized.

① Read!

②



③ Constraint

$x^2 y = 12$

Objective

$$C = \underbrace{2x^2}_{\text{top}} + \underbrace{x^2}_{\text{bottom}} + \underbrace{4xy}_{\text{sides}}$$

$$= 3x^2 + 4xy$$

④ $y = \frac{12}{x^2}$

$$\Rightarrow C = 3x^2 + 4x\left(\frac{12}{x^2}\right)$$

$$= 3x^2 + \frac{48}{x}$$

⑤ $C' = 6x - \frac{48}{x^2} = 0$ ← for crit. pts.

$$\Rightarrow x^3 = \frac{48}{6} = 8$$

$$\Rightarrow x = 2$$

$$\Rightarrow y = \frac{12}{2^2} = 3$$

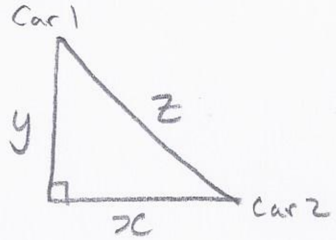
⑥ Dimensions : $\boxed{2 \times 2 \times 3}$

or

$\boxed{\text{length} = \text{width} = 2 \text{ ft, height} = 3 \text{ ft}}$

3. (20 points) Two cars drive away from the same point. One heads north at 40 mph. The other drives east at 30 mph. After 1 hour, at what rate is the distance between them changing?

① Read ☺

② 

x = distance east car drives
y = " north " "
z = " between them

③ know

$\frac{dy}{dt} = 40$

$\frac{dx}{dt} = 30$

want

$\frac{dz}{dt}$ when $x = 30$, $y = 40$

④ $z^2 = x^2 + y^2$

$\Rightarrow \cancel{2}z\frac{dz}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$

⑤ when $x=30, y=40 \Rightarrow z=50$ (3-4-5 Δ)

$\Rightarrow 50\frac{dz}{dt} = (30)(30) + (40)(40)$

$\Rightarrow \frac{dz}{dt} = \frac{\overset{6}{30}(3\cancel{0}) + \overset{8}{40}(4\cancel{0})}{5\cancel{0}}$

$= 18 + 32$

$= \boxed{50 \text{ mph}}$ i.e. the distance between them is increasing at 50 mph.

4. (20 points) Sketch the graph of the function $f(x) = x^3 - 12x^2 + 36x$ by first finding (provided they exist) the domain, intercepts, asymptotes, local extrema, inflection point(s), intervals of increasing and decreasing, and intervals of concavity. The preceding must be indicated on your graph.

① Domain :

$\boxed{(-\infty, \infty)}$

② Intercepts:

x-int: $x^3 - 12x^2 + 36x = 0$

$x(x^2 - 12x + 36) = 0$

$x(x-6)(x-6) = 0$

$\boxed{x = 0, x = 6} \rightarrow$ x-int

y-int: $\boxed{y = 0} \rightarrow$ y-int.

③ Asymptotes: $\boxed{\text{None!}}$

Polynomial.

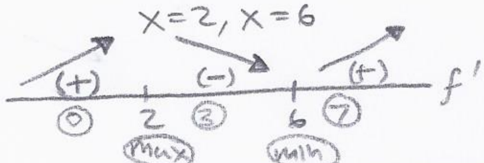
④ Max/min/Inc/Dec

$f' = 3x^2 - 24x + 36 \overset{?}{=} 0$ ← for crit pts

$\Rightarrow 3(x^2 - 8x + 12) = 0$

$\Rightarrow 3(x-2)(x-6) = 0$

$x = 2, x = 6$

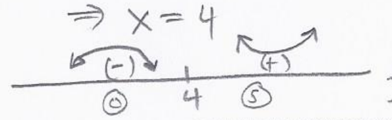


Inc: $(-\infty, 2) \cup (6, \infty)$	max: $(2, 32)$
Dec: $(2, 6)$	min: $(6, 0)$

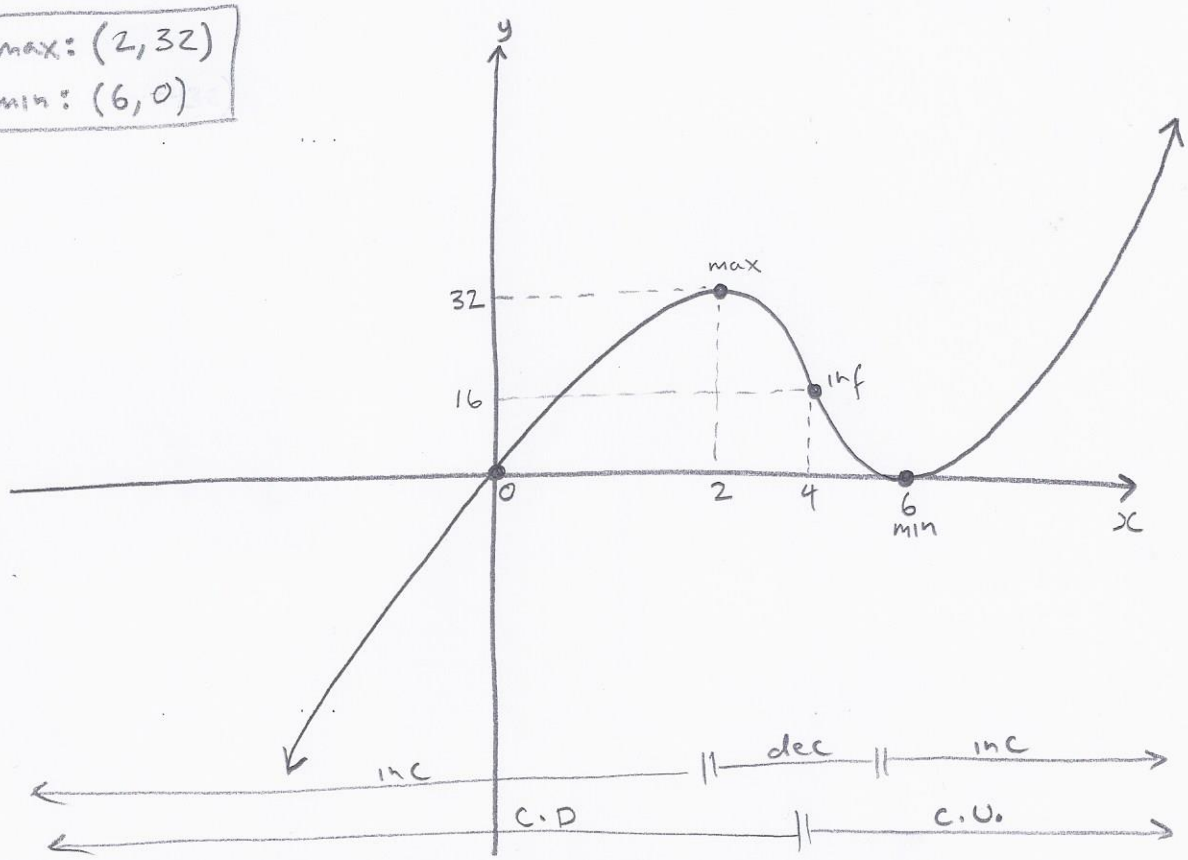
⑤ C.U/C.D./Inf

$f'' = 6x - 24 \overset{?}{=} 0$ ← for crit. pts.

$\Rightarrow x = 4$



C.U: $(4, \infty)$	Inf: $(4, 16)$
C.D: $(-\infty, 4)$	



5. (20 points) Let $f(x) = 2x^3 - 3x^2 - 12x + 1$, find the absolute maximum and minimum of $f(x)$ on the interval $[-2,1]$.

$f' = 6x^2 - 6x - 12$

For crit. pts:

$6x^2 - 6x - 12 = 0$

$\Rightarrow 6(x^2 - x - 2) = 0$

$\Rightarrow 6(x-2)(x+1) = 0$

$\Rightarrow x = -1, \ \cancel{x = 2}$

reject. (Not in interval).

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1 = 8$

End points

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1 = -3$

$f(1) = 2(1)^3 - 3(1)^2 - 12(1) + 1 = -12$

$\Rightarrow$ $\boxed{\begin{array}{l} \text{abs max} = f(-1) = 8 \\ \text{abs min} = f(1) = -12 \end{array}}$

Bonus Problems: (You must complete all problems in the actual test to be eligible).

1. (5 points) Using Riemann sums with 4 subintervals, approximate the area under $f(x) = x^2 + 1$ on the interval $[-1,3]$ using left hand endpoints. Show all your work.

$-1 \quad 0 \quad 1 \quad 2 \quad 3$ $\qquad \Delta x = \frac{3-(-1)}{4} = 1$

$$A \approx \Delta x\left(f(-1)+f(0)+f(1)+f(2)\right)$$
$$= 1\left((-1)^2+1 + 0^2+1 + 1^2+1 + 2^2+1\right)$$
$$= \boxed{10}$$

2. (5 points) Find the exact area under $f(x) = x^2 + 1$ on $[-1,3]$. Is your approximation in problem 1 an over or underestimate?

$$A = \int_{-1}^{3} x^2+1\,dx$$
$$= \frac{x^3}{3} + x \Big|_{-1}^{3}$$
$$= 9+3-\left(\tfrac{-1}{3}-1\right)$$
$$= \boxed{\tfrac{40}{3}}$$

$\boxed{\text{The above is an underestimate!}}$

3. (5 points) The half-life of a radioactive substance is 1200 years. Find and simplify $P(t)$, the amount of substance remaining after t years.

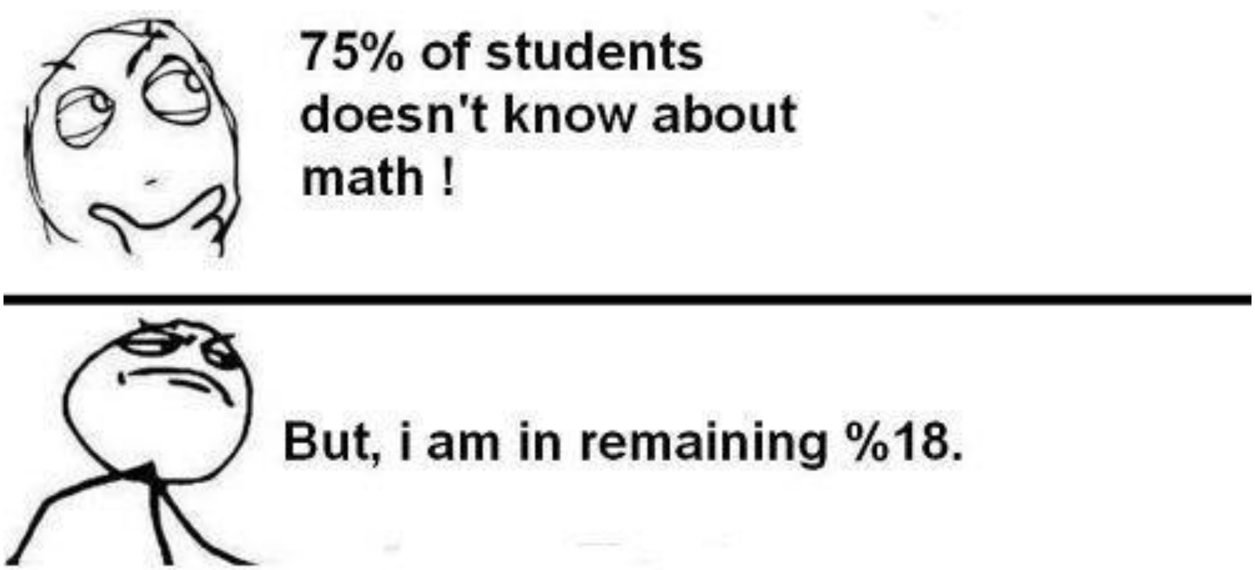
$$r = \frac{\ln 2}{1200}$$
$$\Rightarrow \boxed{P = P_0 e^{-\frac{\ln 2}{1200}t}}$$

4. (5 points) Complete the following formulas.

(a) $\int x^n dx = \boxed{\frac{x^{n+1}}{n+1} + C}$, $n \neq -1$ $\qquad$ (b) $\int 1/x\ dx = \boxed{\ln|x| + C}$

(c) $\int e^{kx}\,dx = \boxed{\frac{1}{k}e^{kx} + C}$, $k \neq 0$.

Always believe in yourself, no matter what.



Math 205 Test 3 Grading Guidelines

Problem 1

Part a: 1 point for figuring out r, 2 points for setting up the equation $P' = \frac{\ln 2}{3} P$ and 1 point for the initial condition.

Part b: All or nothing. This is really just plugging into a formula.

Part c: All or nothing. Again, just plugging into a formula.

Part d: 1 point for knowing to plug in 10 for P. 2 points for calculations. 1 point for the final answer. Be careful how the division signs are used! They should pretty much be exactly as in the solutions. And points will be deducted for cancelling something inside an "ln" with something outside the "ln".

Part e: All or nothing. Another plugging into the right formula.

Problem 2

The variables used may be different, but the diagram should be labeled similarly, with both sides of the base being labeled with the same letter.

2 points to draw and label the diagram.
2 points to figure out the constraint equation.
4 points to figure out the objective equation.
4 points to do step 4.
4 points to find x.
2 points to find y.
2 points to give the final answer of the dimensions.

Problem 3

4 points for properly labeled diagram. Variables may be different, but the idea should be the same. And the sides should NOT be labeled with constants.
4 points to set up the right equation using Pythagoras' theorem
4 points to differentiate implicitly correctly. It is OK to write something like x' instead of $\frac{dx}{dt}$.
2 points to figure out the value of z (remember, a different variable may be used).
4 points to plug in all the numbers in the right positions
2 points for the final answer of 50 mph.

Problem 4

All the required info should be boxed or indicated as they are in the solutions.

2 points for the domain.
3 points to find the x-intercept.

1 point to find the y-intercept.
1 point to state there are no asymptotes.
2 points to find the critical points and do the first derivative test chart.
2 points to state the intervals of increasing and decreasing and max and min points.
2 points to find the second derivative and its critical points.
2 points to state where the graph is concave up, down and where the inflection point is.
5 points for the fully labeled graph.

Problem 5

8 points to differentiate correctly and find the critical points.
2 points to reject $x = 2$.
2 points to evaluate $f(-1)$ correctly.
2 points to evaluate $f(-2)$ correctly.
2 points to evaluate $f(1)$ correctly.
4 points for final answer.

Bonus

1. 1 point to find Δx, 2 points to plug into the formula correctly. 2 points to simplify to 10.

2. 2 points to figure out that a definite integral is needed. 2 points to evaluate the integral. 1 point for the final answer of 40/3.

3. 2 points to figure out r, 3 points to write down the right formula.

4. All or nothing. The student must write the "+C". 1 point for formula (a), 2 points for formula (b) and 2 points for formula (c).

Required knowledge:

For this exam, the student needs to: know algebra (of course!). Also the formulas for exponential growth and decay and how to apply them. See the exponential growth and decay handout. The student also needed to know all the steps to solve an optimization and a related rates problem. The student also needed to know all the steps to do a complete curve sketching problem. There are handouts for optimization, related rates and curve sketching. The student should also be able to find absolute maximums/minimums of a continuous function on a closed interval using the closed interval method.

Optional Knowledge:

To gain bonus points, the student needed to know how to finite Riemann sums and how to find exact areas using definite integrals. Exponential decay formulas were needed here also. Students also needed to know some basic integration formulas.