Math 205 Test 2
July 5, 2016

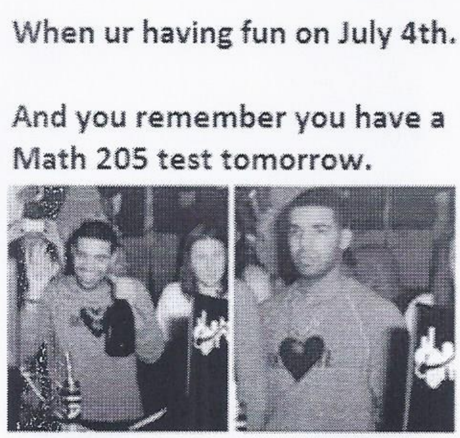
Name: Jhevon Smith

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

Uncle Sam wants YOU to get a good grade on this test.



1. (a) (15 points) Let $f(x) = 2x - \frac{3}{x}$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h) - \frac{3}{x+h} - \left(2x - \frac{3}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h - \frac{3}{x+h} - \cancel{2x} + \frac{3}{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{2\cancel{h}}{\cancel{h}} + \frac{\frac{3}{x} - \frac{3}{x+h}}{h} \right)$$

$$= \lim_{h\to 0} \left(2 + \frac{\frac{3}{x} - \frac{3}{x+h}}{h} \cdot \frac{x(x+h)}{x(x+h)} \right)$$

$$= \lim_{h\to 0} \left(2 + \frac{3(x+h) - 3x}{h \cdot x(x+h)} \right)$$

$$= \lim_{h\to 0} \left(2 + \frac{\cancel{3x} + 3\cancel{h} - \cancel{3x}}{\cancel{h}x(x+h)} \right)$$

$$= \lim_{h\to 0} \left(2 + \frac{3}{x(x+h)} \right)$$

$$= \boxed{2 + \frac{3}{x^2}}$$

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 1$. **Write your line in $y = mx + b$ form.**

$m = f'(1) = 2 + \frac{3}{1^2} = 5$

$y = f(1) = 2(1) - \frac{3}{1} = -1$

tangent line: $y - y_1 = m(x - x_1)$

$\Rightarrow y + 1 = 5(x - 1)$

$\Rightarrow \boxed{y = 5x - 6}$

2. Find $\frac{dy}{dx} = y'$ for the following. Simplify your answers. (4 points each)

(a) $y = \frac{5x^3+3x^2}{2x}$ $= \frac{5x^3}{2x} + \frac{3x^2}{2x}$

$= \frac{5}{2}x^2 + \frac{3}{2}x \longrightarrow$ Power Rule!

$\Rightarrow \boxed{y' = 5x + \frac{3}{2}}$

(b) $y = 2\sqrt{x} + \frac{5}{\sqrt[3]{x}} - \ln(x^2+1)^3 - \pi^2$

$= 2x^{\frac{1}{2}} + 5x^{-1/3} - 3\ln(x^2+1) - \pi^2 \longrightarrow$ Power Rule and log. diff. rule!

$\Rightarrow y' = x^{-1/2} - \frac{5}{3}x^{-4/3} - 3 \cdot \frac{2x}{x^2+1}$

$= \boxed{x^{-1/2} - \frac{5}{3}x^{-4/3} - \frac{6x}{x^2+1}}$

(c) $y = \frac{x^5}{5+x^5}$ $\longrightarrow$ Quotient Rule!

$\Rightarrow y' = \frac{5x^4(5+x^5) - x^5(5x^4)}{(5+x^5)^2}$

$= \frac{5x^4\left[5 + \cancel{x^5} - \cancel{x^5}\right]}{(5+x^5)^2}$

$= \boxed{\frac{25x^4}{(5+x^5)^2}}$

(d) $y = x^2\sqrt{2^x - 1} + x^{x^2}$

One way:

$y = x^2(2^x-1)^{1/2} + e^{\ln x^{x^2}}$

$= x^2(2^x-1)^{1/2} + e^{x^2 \ln x}$

$\Rightarrow y' = 2x(2^x-1)^{1/2} + x^2 \cdot \frac{1}{2}(2^x-1)^{-1/2}(2^x \ln 2) + (2x\ln x + x)e^{x^2 \ln x}$

$= \boxed{2x(2^x-1)^{1/2} + x^2(2^x-1)^{-1/2}2^{x-1}\ln 2 + (2x\ln x + x)x^{x^2}}$

(See next page for another way to do this!)

2. Find $\frac{dy}{dx} = y'$ for the following. Simplify your answers. (4 points each)

(a) $y = \frac{5x^3+3x^2}{2x}$

(b) $y = 2\sqrt{x} + \frac{5}{\sqrt[3]{x}} - \ln(x^2+1)^3 - \pi^2$

(c) $y = \frac{x^5}{5+x^5}$

(d) $y = x^2\sqrt{2^x - 1} + x^{x^2}$

Another way:

First do log diff. for

$g = x^{x^2}$

$\Rightarrow \ln g = \ln x^{x^2}$

$\Rightarrow \ln g = x^2 \ln x$

$\Rightarrow \frac{g'}{g} = 2x\ln x + x$

$\Rightarrow g' = g(2x\ln x + x)$

$= x^{x^2}(2x\ln x + x)$

$\Rightarrow y' = 2x(2^x-1)^{1/2} + x^2 \frac{1}{2}(2^x-1)^{-1/2}(2^x \ln 2)$

$+$

$x^{x^2}(2x\ln x + x)$

$$\Rightarrow \boxed{y' = 2x(2^x-1)^{1/2} + x^2(2^x-1)^{-1/2}\, 2^{x-1}\ln 2 + x^{x^2}(2x\ln x + x)}$$

(e) $x^2y + 2x + 3y = 7x + 12$

$\Rightarrow 2xy + x^2y' + 2 + 3y' = 7$

$\Rightarrow y'(x^2+3) = 7 - 2xy - 2$

$\Rightarrow \boxed{y' = \dfrac{5-2xy}{x^2+3}}$

3. (5 points each part) A bunch of angry calculus students (allegedly) throw Jhevon off a cliff. Jhevon's position above the ground at time t seconds is given by $s(t) = -16t^2 + 16t + 96$.

(a) Find functions that describe Jhevon's velocity and acceleration at time t.

Velocity function $= \boxed{v(t) = -32t + 16}$

Acceleration function $= \boxed{a(t) = -32}$

(b) What is the highest height that Jhevon attains?

One way:

Find vertex of $s(t)$:

$t = \frac{-b}{2a} = \frac{-16}{2(-16)} = \frac{1}{2}$ second.

$\Rightarrow$ max height $= s(\frac{1}{2})$

$= -16(\frac{1}{2})^2 + 16(\frac{1}{2}) + 96$

$= -4 + 8 + 96$

$= \boxed{100 \text{ ft}}$

Another way:

Max height $\Rightarrow v(t) = 0$

$\Rightarrow -32t + 16 = 0$

$\Rightarrow t = 1/2$

$\Rightarrow$ max height $= s(\frac{1}{2})$

$= \boxed{100 \text{ ft}}$

(c) When will Jhevon hit the ground and the nightmare end for his students?

We want when $s(t) = 0$

$\Rightarrow -16t^2 + 16t + 96 = 0$

$\Rightarrow t^2 - t - 6 = 0$

$\Rightarrow (t-3)(t+2) = 0$

$\Rightarrow \boxed{t = 3}$, $t = -2$

reject! Negative time.

$\Rightarrow$ Jhevon hits the ground after $t = 3$ seconds

(d) With what velocity will Jhevon hit the ground? This number shall be commemorated with fond memories.

We want $v(3) = -32(3) + 16$

$= -96 + 16$

$\boxed{= -80 \text{ ft/sec}}$

4. (i) (2 points each) State the following rules precisely:

(a) The power rule for derivatives: $\frac{d}{dx} x^n = nx^{n-1}$

(b) The chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

(c) The quotient rule: $\frac{d}{dx} \frac{f}{g} = \frac{f'g - fg'}{g^2}$

(d) The product rule: $\frac{d}{dx}(f \cdot g) = f'g + fg'$

(f) The rule to differentiate a general exponential with base a and power u. $\frac{d}{dx} a^u = u' a^u \ln a$

(g) The rule to differentiate the natural logarithm of a function u. $\frac{d}{dx} \ln u = \frac{u'}{u}$

(ii) (8 points) Use linear approximation to approximate $\sqrt[3]{27.1}$. You may leave your answer as a sum of fractions.

Set $f(x) = \sqrt[3]{x} = x^{1/3}$

Set $x = 27.1$, $a = 27$

$\Rightarrow f(x) = x^{1/3} \Rightarrow f(a) = 27^{1/3} = 3$

$\Rightarrow f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(a) = \frac{1}{3} \cdot 27^{-2/3} = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$

$\Rightarrow \sqrt[3]{27.1} = f(x) \approx f(a) + f'(a)(x-a)$

$= 3 + \frac{1}{27}(27.1 - 27)$

$= 3 + \frac{1}{27} \cdot \frac{1}{10}$

$= \boxed{3 + \frac{1}{270}}$

5. We return to our story, where our hero, Jhevon, is trying to get his hotdog business to be the best in the country. The cost $C(x)$, in dollars, of producing x hotdogs is given by

$$C(x) = 50 - 20x + 2x^2$$

Assuming Jhevon will sell only specialty hotdogs at \$5/hotdog, answer the following:

(a) (4 points) What is Jhevon's revenue function, $R(x)$, and profit function, $P(x)$?

$\boxed{R(x) = 5x}$, $P(x) = R(x) - C(x)$

$\Rightarrow \boxed{P(x) = 5x - (50 - 20x + 2x^2)}$ or $\boxed{P(x) = -50 + 25x - 2x^2}$

(b) (4 points) Find the marginal cost and marginal revenue functions.

$\boxed{C'(x) = -20 + 4x}$ Marginal cost

$\boxed{R'(x) = 5}$ Marginal Revenue

(c) (4 points) Assume Jhevon made 6 hotdogs, use the marginal cost to approximate what it would cost him to make the seventh.

Cost for the $7^{th} \approx C'(6)$

$= -20 + 4(6)$

$\boxed{= \$4}$

(d) (6 points) How many hotdogs does Jhevon need to sell to break even?

Break Even $\Rightarrow R(x) = C(x)$

$\Rightarrow 5x = 50 - 20x + 2x^2$

$\Rightarrow 2x^2 - 25x + 50 = 0$

$\Rightarrow (2x - 5)(x - 10) = 0$

$x = 5/2$, $\boxed{x = 10}$

reject

(e) (2 points) Considering all of the above, is the current situation good for business? Justify.

No! Profit is an inverted parabola!

Eventually, profit will decline with more production. Cost will grow faster than revenue.

Stay tuned for the next exciting episode of Jhevon's Hotdog Stand

Bonus Problems: (You must complete all problems in the actual test to be eligible).

1. (5 points each) For problem 2(e):

   (i) Find $\frac{dy}{dt}$

$$2x\frac{dx}{dt}y + x^2\frac{dy}{dt} + 2\frac{dx}{dt} + 3\frac{dy}{dt} = 7\frac{dx}{dt}$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{(5-2xy)\frac{dx}{dt}}{x^2+3}}$$

   (ii) Find $\frac{dx}{dy}$

$$\Rightarrow 2x\frac{dx}{dy}y + x^2 + 2\frac{dx}{dy} + 3 = 7\frac{dx}{dy}$$

$$\Rightarrow (2xy-5)\frac{dx}{dy} = -(x^2+3)$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{x^2+3}{5-2xy}}$$

2. (5 points) The concentration of a drug in a patient's bloodstream t hours after it is taken is given by

$$C(t) = \frac{0.016t}{(t+2)^2}\, mg/cm^3.$$

Find the maximum concentration of the drug and the time at which it occurs.

Max will occur where $C'(t) = 0$ or undefined.

$$C'(t) = \frac{0.016(t+2)^2 - 0.016t \cdot 2(t+2)}{(t+2)^4}$$

$$= \frac{0.016\cancel{(t+2)}[t+2-2t]}{(t+2)^{\cancel{4}3}}$$

Crit. pts: $2-t=0$ and $t=-2$

$t=2$ reject.

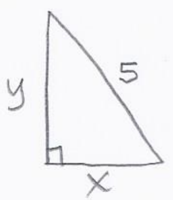
→ Max concentration occurs when $\boxed{t=2 \text{ hrs}}$

Its value is

$$C(2) = \frac{0.016(2)}{(2+2)^2}$$

$$= \frac{0.016(2)}{16} = \frac{2}{1000} = \boxed{\frac{1}{500}\frac{mg}{cm^3}}$$

3. (5 points) A 5 foot ladder leans against a vertical wall. Batman pushes the foot of the ladder towards the wall at a rate of 2 ft/sec. At what rate is top of the ladder moving along the wall when the foot of the ladder is 3 feet from the wall? Include a sketch in your answer.



$\frac{dx}{dt} = -2$

Want $\frac{dy}{dt}$ when $x=3$

By Pythagoras

$$x^2+y^2=5^2$$

$$\Rightarrow \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$$

when $x=3, y=4$

$$\Rightarrow (3)(-2) + 4\frac{dy}{dt} = 0$$

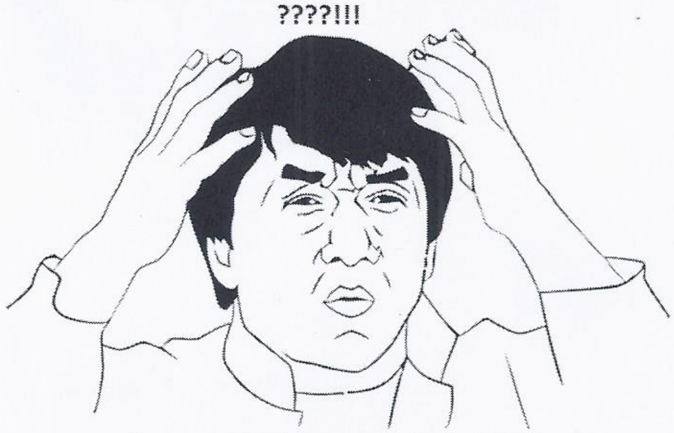
$$4\frac{dy}{dt} = 6$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{3}{2}}\ ft/sec$$

Differentiate

$$y = x^2\sqrt{2^x - 1} + x^{x^2}$$

????!!!



Jhevon, what's wrong witchu???!!!