MATH 203 TEST 3B

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Name: JHEVON SMITH

Note that both sides of each sheet has printed material

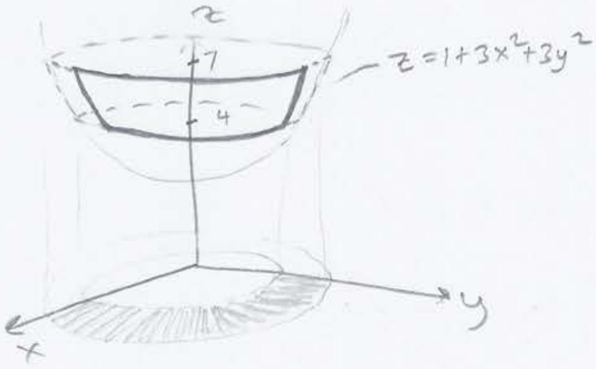
Instructions:

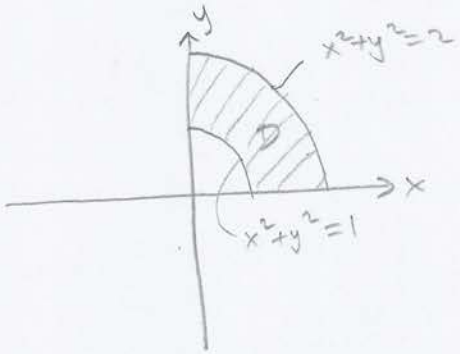
1. Read the instructions.
2. Complete all problems!
3. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answer.
4. Write neatly, so that I am able to follow your sequence of steps, and **box your answers**.
5. Read through the exam and kill all the easy problems (for you) first!
6. No calculators, notes, or other outside aids allowed–including the smart kid that may be sitting beside you, or the friend you were thinking of texting.
7. Use correct notation! Write what you mean! "x^2" and "$x2$" are NOT the same thing, and use the right brackets for vectors (don't forget the commas!), for examples.
8. Other than that, have fun, and good luck!

Remember: The force will be with you...always.

1. Let S be the part of the surface $z = 1 + 3x^2 + 3y^2$ between the planes $z = 4$ and $z = 7$ in the first octant.

(20 points) Compute the surface area of S. Include a sketch in your answer.





$z = f = 1 + 3x^2 + 3y^2$

$\Rightarrow f_x = 6x, \ f_y = 6y$

when $z = 7$: $7 = 1 + 3x^2 + 3y^2 \Rightarrow x^2 + y^2 = 2 \rightarrow r = \sqrt{2}$

when $z = 4$: $4 = 1 + 3x^2 + 3y^2 \Rightarrow x^2 + y^2 = 1 \rightarrow r = 1$

$$\Rightarrow A = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 36x^2 + 36y^2}\, dA$$

Switch to polar coordinates

$$= \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \sqrt{1 + 36r^2}\, r\, dr\, d\theta$$

$u = 1 + 36r^2$, $du = 72r\, dr$ | when $r = \sqrt{2}$, $u = 73$; when $r = 1$, $u = 37$

$$= \frac{1}{72} \int_0^{\frac{\pi}{2}} \int_{37}^{73} u^{1/2}\, du\, d\theta$$

$$= \frac{1}{72} \cdot \frac{2}{3} \int_0^{\pi/2} u^{3/2} \Big|_{37}^{73} d\theta$$

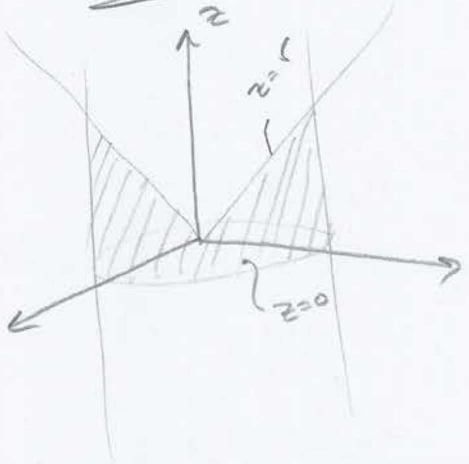
$$= \frac{1}{108} \int_0^{\pi/2} \left(73^{3/2} - 37^{3/2}\right) d\theta$$

$$\boxed{= \frac{\pi}{216}\left(73^{3/2} - 37^{3/2}\right)}$$

2. Let T_1 be the region below the cone $z = \sqrt{x^2 + y^2}$, inside the cylinder $x^2 + y^2 = 1$, above the $xy-$plane. Let T_2 be the region inside the sphere $x^2 + y^2 + z^2 = 1$, but below the cone $z = \sqrt{x^2 + y^2}$.

(a) (20 points) Set up $\iiint_{T_1} (x^2+y^2+z^2)\, dV$ and $\iiint_{T_2} (x^2+y^2+z^2)\, dV$ using coordinate systems of your choice. Justify your answer!

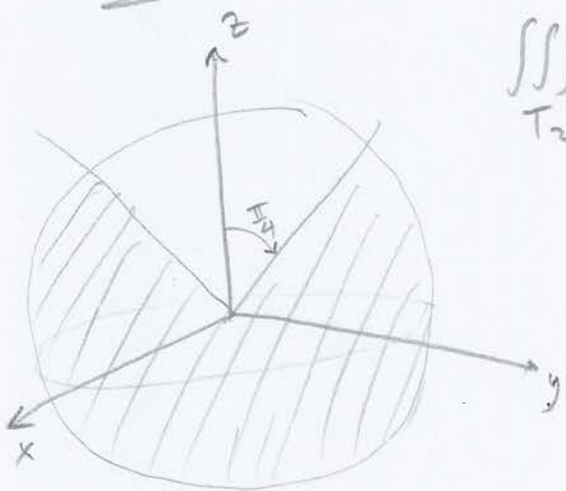
T_1



By Polar

$$\iiint_{T_1} x^2+y^2+z^2 \, dV = \int_0^{2\pi}\int_0^1\int_0^r (r^2+z^2)\, r \, dz\, dr\, d\theta$$

T_2



By Spherical

$$\iiint_{T_2} x^2+y^2+z^2 \, dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^1 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

(b) (20 points) Evaluate either of the integrals set up in 2(a).

$\underline{\underline{I_1}}$

$$\int_0^{2\pi}\int_0^1\int_0^r (r^2+z^2)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 z + \frac{r z^3}{3}\Big|_0^r \,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^4 + \frac{r^4}{3}\,dr\,d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta$$

$$= \frac{4}{15}\int_0^{2\pi} r^5\Big|_0^1\,d\theta$$

$$\boxed{= \frac{8\pi}{15}}$$

$\underline{\underline{OR}}$

$\underline{\underline{I_2}}$

$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^1 \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi} \frac{\rho^5}{5}\sin\phi\Big|_0^1\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi} \frac{1}{5}\sin\phi\,d\phi\,d\theta$$

$$= 2\pi\left(-\frac{1}{5}\cos\phi\right)\Big|_{\frac{\pi}{4}}^{\pi}$$

$$= 2\pi\left(\frac{1}{5}+\frac{\sqrt{2}}{10}\right)$$

$$\boxed{= \frac{2+\sqrt{2}}{5}\pi}$$

3. (10 points) Write down the formulas for the mass and center of mass of an object whose density at the point (x, y, z) is given by $\rho(x, y, z)$.

$$\boxed{m = \iiint_E \rho(x,y,z)\,dV}$$

$$\bar{x} = \iiint_E \frac{x\,\rho(x,y,z)\,dV}{m}$$

$$\bar{y} = \iiint_E \frac{y\,\rho(x,y,z)\,dV}{m}$$

$$\bar{z} = \iiint_E \frac{z\,\rho(x,y,z)\,dV}{m}$$

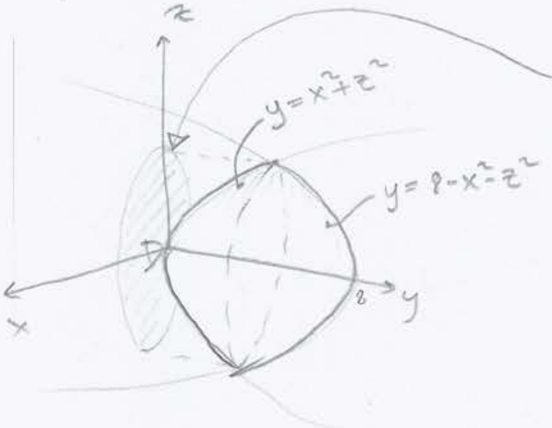
$$\boxed{\text{C.O.M} = (\bar{x}, \bar{y}, \bar{z})}$$

4. (30 points) Find the volume of the region bounded by $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$. Include a sketch.

Intersections...

$x^2 + z^2 = 8 - x^2 - z^2$

$\Rightarrow x^2 + z^2 = 4$



$$\text{Now } V = \iiint_E 1\,dV \quad \text{or} \quad \iint_D \text{"top"} - \text{"bottom"}\,dA$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} 1 \cdot r\,dy\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[(8-r^2) - r^2\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (8r - 2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} 4r^2 - \tfrac{1}{2}r^4 \Big|_0^2\,d\theta$$

$$= \int_0^{2\pi} 8\,d\theta$$

$$\boxed{= 16\pi}$$

Bonus 1: (10 points) For each of the series below, state whether the series is convergent (in any sense). Justify your answer.

(a) $\sum_{n=2}^{\infty} \frac{2n-5}{n^3-4n+3}$ (b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

$\frac{2n-5}{n^3-4n+3} \sim \frac{2n}{n^3}$ for large n.

$= \frac{2}{n^2}$.

Let $a_n = \frac{2n-5}{n^3-4n+3}$, $b_n = \frac{2}{n^2}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$

Hence $\sum a_n$ and $\sum b_n$ both converge or diverge together. By the limit comparison test.

Since $\sum b_n$ converges (convergent p-series),

$\boxed{\sum a_n \text{ also converges!}}$

$a_n = \frac{1}{n\sqrt{\ln n}}$.

Clearly $a_n \geq 0$ and is decreasing (constant numerator, growing denominator).

a_n is also continuous.

Hence, by the integral test: $\sum a_n$ converges iff $\int_2^\infty a_x \, dx$ converges

But $\int_2^\infty a_x \, dx = \int_2^\infty \frac{1}{x\sqrt{\ln x}} dx$

$= \lim_{N\to\infty} \int_2^N \frac{1}{x\sqrt{\ln x}} dx$

$= \lim_{N\to\infty} 2\sqrt{\ln x}\Big|_2^N$

$= \lim_{N\to\infty} \left(2\sqrt{\ln N} - 2\sqrt{\ln 2}\right)$

$= \infty$

$\boxed{\text{Hence } \sum a_n \text{ diverges!}}$

Bonus 2: (5 points) Write down the Maclaurin series for $f(x) = \sin x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$