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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.
11. Other than that, have fun and good luck!

GET 115 ON THIS TEST!!



CHALLENGE ACCEPTED!!

1. Find y' for the following (10 points each):

(a) $y = x^{x^2} + 2^{\sin x}$

$$y = e^{x^2 \ln x} + 2^{\sin x}$$

$$\Rightarrow y' = (2x \ln x + x) x^{x^2} + \cos x \cdot 2^{\sin x} \cdot \ln 2$$

$$= \boxed{(2 \ln x + 1) x^{x^2+1} + \cos x \cdot 2^{\sin x} \cdot \ln 2}$$

OR Set $g(x) = x^{x^2}$

$$\Rightarrow \ln g = x^2 \ln x$$

$$\Rightarrow \frac{g'}{g} = 2x \ln x + x$$

$$\Rightarrow g' = g(2x \ln x + x)$$

$$= x^{x^2} (2x \ln x + x)$$

$$= x^{x^2+1} (2 \ln x + 1)$$

$$\Rightarrow y' = g' + \frac{d}{dx} 2^{\sin x}$$

$$= \boxed{(2 \ln x + 1) x^{x^2+1} + \cos x \cdot 2^{\sin x} \cdot \ln 2}$$

(b) $y = \tan^{-1}(e^{x^2}) + x^2 \ln \cos x$

$$y' = \frac{2x e^{x^2}}{1 + (e^{x^2})^2} + 2x \ln \cos x - x^2 \frac{\sin x}{\cos x}$$

$$= \boxed{\frac{2x e^{x^2}}{1 + e^{2x^2}} + 2x \ln \cos x - x^2 \tan x}$$

$$\begin{aligned}
 & (c) \ln \sqrt{x+y} = x^2 + e^{\pi^2} - \sin(x+y) \\
 \Rightarrow & \frac{1}{2} \ln(x+y) = x^2 + e^{\pi^2} - \sin(x+y) \\
 \Rightarrow & \frac{1}{2} \cdot \frac{1+y'}{x+y} = 2x - (1+y')\cos(x+y) \\
 \Rightarrow & \frac{1}{2(x+y)} + \frac{y'}{2(x+y)} = 2x - \cos(x+y) - y'\cos(x+y) \\
 \Rightarrow & y' \left(\frac{1}{2(x+y)} + \cos(x+y) \right) = 2x - \cos(x+y) - \frac{1}{2(x+y)} \\
 \Rightarrow & y' = \frac{2x - \cos(x+y) - \frac{1}{2(x+y)}}{\frac{1}{2(x+y)} + \cos(x+y)} \\
 = & \boxed{\frac{4x(x+y) - 2(x+y)\cos(x+y) - 1}{1 + 2(x+y)\cos(x+y)}}
 \end{aligned}$$

2. Evaluate the following integrals (10 points each):

$$(a) \int \frac{\sin(3\sqrt{x})}{2\sqrt{x}} dx$$

$$\begin{aligned}
 u &= 3\sqrt{x} \\
 \Rightarrow du &= \frac{3}{2\sqrt{x}} dx \\
 \Rightarrow \frac{1}{3} du &= \frac{1}{2\sqrt{x}} dx
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \frac{1}{3} \int \sin u du \\
 &= -\frac{1}{3} \cos u + C \\
 &= \boxed{-\frac{1}{3} \cos(3\sqrt{x}) + C}
 \end{aligned}$$

$$(b) \int_1^e \frac{\sqrt[3]{\ln x}}{3x} dx$$

$u = \ln x \rightarrow$

$$\Rightarrow du = \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{3} \int_0^1 u^{1/3} du$$

$$= \frac{1}{3} \cdot \frac{3}{4} u^{4/3} \Big|_0^1$$

$$= \boxed{\frac{1}{4}}$$

Changing limits
when $x=e$
 $u=1$
when $x=1$
 $u=0$

OR

$$u^3 = \ln x$$

$$\Rightarrow 3u^2 du = \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{3} \int_0^1 u \cdot 3u^2 du$$

$$= \int_0^1 u^3 du$$

$$= \frac{u^4}{4} \Big|_0^1$$

$$= \boxed{\frac{1}{4}}$$

We could also not change the limits, back-substitute and plug in the old limits into x .

$$(c) \int \frac{\pi^e x^2}{\sqrt[3]{4-2x^3}} dx$$

$$u = 4 - 2x^3$$

$$du = -6x^2 dx$$

$$\Rightarrow -\frac{1}{6} du = x^2 dx$$

$$\Rightarrow -\frac{\pi^e}{6} \int u^{-1/3} du$$

$$= -\frac{\pi^e}{6} \cdot \frac{3}{2} u^{2/3} + C$$

$$= \boxed{-\frac{\pi^e}{4} (4-2x^3)^{2/3} + C}$$

OR

$$u^3 = 4 - 2x^3$$

$$\Rightarrow 3u^2 du = -6x^2 dx$$

$$\Rightarrow -\frac{1}{2} u^2 du = x^2 dx$$

$$\Rightarrow -\frac{\pi^e}{2} \int \frac{1}{u} \cdot u^2 du$$

$$= -\frac{\pi^e}{2} \cdot \frac{u^2}{2} + C$$

$$= -\frac{\pi^e}{4} u^2 + C$$

$$= \boxed{-\frac{\pi^e}{4} (4-2x^3)^{2/3} + C}$$

3. Compute the following limits (10 points each):

$$\begin{aligned}(a) \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{x-2} \\&= \lim_{x \rightarrow 2} (x+1) \\&= \boxed{3}\end{aligned}$$

$$(b) \lim_{x \rightarrow 0} \frac{e^x - (1-x)}{x^2} \rightarrow \frac{0}{0}$$

Apply L'H

$$= \lim_{x \rightarrow 0} \frac{e^x + 1}{2x} \rightarrow \frac{2}{0}, \text{ can't use L'H!}$$

Note that

$$\lim_{x \rightarrow 0^-} \frac{e^x + 1}{2x} = -\infty \text{ while } \lim_{x \rightarrow 0^+} \frac{e^x + 1}{2x} = +\infty$$

$\Rightarrow \boxed{\text{limit D.N.E.}}$

$$\begin{aligned}
 & (c) \lim_{x \rightarrow 0^+} (1 + 3x)^{\frac{2}{x}} \\
 & = \lim_{x \rightarrow 0^+} e^{\frac{2 \ln(1+3x)}{x}} \quad \text{Let } z = \frac{2 \ln(1+3x)}{x} \rightarrow \frac{0}{0} \\
 & = \lim_{x \rightarrow 0^+} e^{z} \\
 & L'H \quad = \lim_{x \rightarrow 0^+} e^{\frac{\frac{6}{1+3x}}{1}} \\
 & = \boxed{e^6}
 \end{aligned}$$

OR

$$\begin{aligned}
 & \text{Let } y = \lim_{x \rightarrow 0^+} (1 + 3x)^{\frac{2}{x}} \\
 & \Rightarrow \ln y = \ln \left(\lim_{x \rightarrow 0^+} (1 + 3x)^{\frac{2}{x}} \right) \\
 & = \lim_{x \rightarrow 0^+} \ln(1 + 3x)^{\frac{2}{x}} \text{ by continuity} \\
 & = \lim_{x \rightarrow 0^+} \frac{2 \ln(1 + 3x)}{x} \rightarrow \frac{0}{0} \\
 & L'H \quad = \lim_{x \rightarrow 0^+} \frac{\frac{6}{1+3x}}{1} \\
 & \Rightarrow \ln y = 6 \\
 & \Rightarrow \boxed{y = e^6}
 \end{aligned}$$

4. A population of bacteria, exhibiting exponential growth, doubles every four hours. If you start with 20 members of such a population:

(a) What is the relative growth rate? (2 points)

$$\begin{aligned}
 & r = \frac{\ln 2}{4} \\
 & (b) \text{ Find a formula, } P(t), \text{ for the size of the population at time } t. \text{ (1 point)}
 \end{aligned}$$

(c) After how long will the population have 36 members? (3 points)

$$\begin{aligned}
 & 36 = 20 e^{\frac{\ln 2}{4} t} \\
 & \Rightarrow \frac{36}{20} = e^{\frac{\ln 2}{4} t} \\
 & \Rightarrow \frac{9}{5} = e^{\frac{\ln 2}{4} t} \\
 & \Rightarrow t = \frac{\ln \frac{9}{5}}{\frac{\ln 2}{4}} \\
 & \Rightarrow \boxed{t = \frac{4 \ln \frac{9}{5}}{\ln 2}} \quad \text{hours}
 \end{aligned}$$

(d) What is the rate of growth when the population is 36? (2 points)

$$\begin{aligned} P' &= rP \\ \Rightarrow P' &= \frac{\ln 2}{4} \cdot 36 \\ \Rightarrow P' &= 9 \ln 2 \end{aligned}$$

(e) What will be the size of the population after 13 hours? (2 points)

$$\begin{aligned} P(13) &= 20e^{\frac{\ln 2}{4}(13)} \\ &= 20e^{\frac{13 \ln 2}{4}} \quad \text{or} \quad 20 \cdot 2^{\frac{13}{4}} \end{aligned}$$

Bonus Problems (5 points each): Evaluate each integral; use a trig substitution to do part (c) (you should actually know the answer to that one by heart, so you'll know if you get the answer correct).

$$(a) \int \frac{2x^3}{\sqrt{4-x^2}} dx$$

$$\begin{aligned} u^2 &= 4 - x^2 \\ \Rightarrow 2u du &= -2x dx \\ \Rightarrow x^2 &= 4 - u^2 \\ \Rightarrow -2 \int \frac{4-u^2}{u} \cdot u du & \\ = -2 \int 4-u^2 du & \\ = -2 \left(4u - \frac{u^3}{3} \right) + C & \\ = \boxed{-8\sqrt{4-x^2} + \frac{2}{3}(4-x^2)^{3/2} + C} & \end{aligned}$$

OR

$$\begin{aligned} u &= 4 - x^2 \\ \Rightarrow x^2 &= 4 - u \\ \Rightarrow du &= -2x dx \\ \Rightarrow -\frac{1}{2} du &= x dx \\ \Rightarrow -\int \frac{4-u}{u^{1/2}} du & \\ = -\int 4u^{-1/2} - u^{1/2} du & \\ = -\left(8u^{1/2} - \frac{2}{3}u^{3/2} \right) + C & \\ = \boxed{-8\sqrt{4-x^2} + \frac{2}{3}(4-x^2)^{3/2} + C} & \end{aligned}$$

$$(b) \int \frac{\sin^3(\pi^2 x)}{\sqrt[3]{\cos(\pi^2 x)}} dx$$

$$= \int \frac{(1 - \cos^2(\pi^2 x)) \sin(\pi^2 x)}{\sqrt[3]{\cos(\pi^2 x)}} dx$$

$$\begin{aligned} u^3 &= \cos(\pi^2 x) \\ \Rightarrow 3u^2 du &= -\pi^2 \sin \pi^2 x dx \\ \Rightarrow -\frac{3u^2}{\pi^2} du &= \sin \pi^2 x dx \\ = -\frac{3}{\pi^2} \int \frac{(1-u^6) \cdot u^2}{u} du & \\ = -\frac{3}{\pi^2} \int u - u^7 du & \\ = -\frac{3}{\pi^2} \left(\frac{u^2}{2} - \frac{u^8}{8} \right) + C & \\ = \boxed{-\frac{3}{\pi^2} \left[\frac{(\cos(\pi^2 x))^{2/3}}{2} - \frac{(\cos \pi^2 x)^{8/3}}{8} \right] + C} & \end{aligned}$$

$$(c) \int \frac{1}{\sqrt{1-x^2}} dx$$

$$x = \sin \theta$$

$$\Rightarrow dx = \cos \theta d\theta$$

$$\Rightarrow \int \frac{1}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta d\theta$$

$$= \int \frac{\cos \theta}{\cos \theta} d\theta$$

$$= \int 1 d\theta$$

$$= \theta + C$$

$$= \boxed{\sin^{-1} x + C}$$

GOT 100% Yes...

