Math 201 Test 1B

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems. In this exam, each non-bonus problem is worth 20 points. The weight of the bonus problems are indicated.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**
8. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
9. Other than that, have fun and good luck!

Remember: math is fun, math is beautiful, this test is *not* hard, there is no spoon.

1. (a) Compute the following limits, **you need not show your work for part (a)**, just state the answer. Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to-\infty} \frac{x^3 \cos x^2}{4-x^3} =$ DNE

(ii) $\lim_{x\to-\infty} \frac{(2x+4)^3(2-3x^2)}{(3-x^2)(2x+1)^3} =$ 3

(iii) $\lim_{x\to\infty} \frac{7-\frac{2}{x}+\frac{3\pi}{x^2}-x^{10}+\frac{2x^7}{5}}{2\pi x^{10}+6x^3-\pi x^9} =$ $-\frac{1}{2\pi}$

(iv) $\lim_{x\to-\infty} \frac{\pi^2+3x^2-3\pi x^4}{4x^3+3\pi x+2} =$ $+\infty$

(b) Compute the following limits. **Show your work in this part**. Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to 0} \frac{\tan 4x+2x}{\sin 3x}$

$$= \lim_{x\to 0} \frac{\frac{\tan 4x}{4x} + \frac{2x}{4x}}{\frac{\sin 3x}{4x}\cdot\frac{3}{3}} = \frac{1+\frac{1}{2}}{3/4} = \frac{3}{2}\cdot\frac{4}{3}$$

$$\boxed{= 2}$$

(i) $\lim_{x\to -1} \frac{x^2-4x}{|x^2-3x-4|}$

$$= \lim_{x\to -1} \frac{x(x-4)}{|x-4||x+1|} = \frac{5}{0^+}$$

$$\boxed{= +\infty}$$

(ii) $\lim_{x\to -2} (x^2+x-2)\sin^2\frac{1}{(x+2)^2}$

Note that, if $x \neq -2$, $0 \le \sin^2\frac{1}{(x+2)^2} \le 1$

$$\Rightarrow 0 \le (x^2+x-2)\sin^2\frac{1}{(x+2)^2} \le x^2+x-2$$

$$\Rightarrow 0 \le \lim_{x\to -2}(x^2+x-2)\sin^2\frac{1}{(x+2)^2} \le \lim_{x\to -2}(x^2+x-2) = 0$$

$$\Rightarrow \boxed{\lim_{x\to -2}(x^2+x-2)\sin^2\frac{1}{(x+2)^2} = 0}$$

by the Squeeze Theorem

2. Let $f(x) = 4 + 2x - x^2$ and $g(x) = \sqrt{9 - x^2}$. Find and simplify

(i) $f \circ g = 4 + 2\sqrt{9-x^2} - (\sqrt{9-x^2})^2$

$$\boxed{= x^2 + 2\sqrt{9-x^2} - 5}$$

(ii) $g \circ f = \sqrt{9 - (4+2x-x^2)^2}$

$$= \sqrt{9 - 16 - 8x + 4x^2 - 8x - 4x^2 + 2x^3 + 4x^2 + 2x^3 - x^4}$$

$$\boxed{= \sqrt{-x^4 + 4x^3 + 4x^2 - 16x - 7}}$$

(iii) $g \circ g$

$$= \sqrt{9 - (\sqrt{9-x^2})^2}$$

$$= \sqrt{x^2} = \boxed{|x|}$$

(iv) $f \circ f$

$$= 4 + 2(4+2x-x^2) - (4+2x-x^2)^2$$

$$= 4 + 8 + 4x - 2x^2 - x^4 + 4x^3 + 4x^2 - 16x - 16$$

$$\boxed{= -x^4 + 4x^3 + 2x^2 - 12x - 4}$$

(v) $\frac{f(x+h)-f(x)}{h} = \frac{4 + 2(x+h) - (x+h)^2 - (4+2x-x^2)}{h}$

$$= \frac{\cancel{4} + \cancel{2x} + 2h - \cancel{x^2} - 2xh - h^2 - \cancel{4} - \cancel{2x} + \cancel{x^2}}{h}$$

$$\boxed{= 2 - 2x - h}$$

3. Use the intermediate value theorem to show that there is a root of the equation $\cos x = x^2$ in the interval (0,1). What condition(s) allow you to use the intermediate value theorem here?

Let $f(x) = \cos x - x^2$. We need to show $f(x) = 0$ for some $x \in (0,1)$.

$\boxed{\text{Since } f \text{ is continuous, we can use the IVT}}$.

Note that $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 < 0$.

Since $f(1) < 0 < f(0)$, there exists $c \in (0,1)$ such that $f(c) = 0$. Done!

4. (a) State an equation that defines what it means for a function $q(x)$ to be continuous at a point $x = c$.

$$\boxed{\lim_{x \to c} q(x) = q(c)}$$

(b) State the intervals where the following functions are continuous. Justify your claim!

(i) $f(x) = 12 + 4x + 7x^{11}$

$\boxed{\text{Continuous on } (-\infty, \infty).}$ Polynomial!

(ii) $m(x) = \begin{cases} x^4 \cos^2\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Note, for $x \neq 0$, $0 \leq x^4 \cos^2\left(\frac{1}{x}\right) \leq x^4$

$\Rightarrow \lim_{x \to 0} x^4 \cos^2\left(\frac{1}{x}\right) = 0$

by the Squeeze Theorem

$\Rightarrow \boxed{\text{continuous on } (-\infty, \infty).}$

(iii) $g(x) = \frac{2x^2}{\sqrt{2-x} - \sqrt{3+x}}$

$\sqrt{2-x} - \sqrt{3+x} = 0$

$\Rightarrow \sqrt{2-x} = \sqrt{3+x}$

$\Rightarrow 2 - x = 3 + x$

$\Rightarrow x = -\frac{1}{2}$

also need $2 - x \geq 0$

$\Rightarrow x \leq 2$

and $3 + x \geq 0$

$\Rightarrow x \geq -3$

$\Rightarrow \boxed{\text{continuous on } [-3, -\tfrac{1}{2}) \cup (-\tfrac{1}{2}, 2]}$

5. Find all possible values of the constants a, b, c and d that make the following functions continuous everywhere.

(a) $f(x) = \begin{cases} 2a - x, & x \leq -1 \\ \frac{x^2+2x+b}{x+1}, & x > -1 \end{cases}$

For $x > -1$, we need $b = 1$,

so that $\frac{x^2+2x+b}{x+1} = \frac{(x+1)^2}{x+1} = x + 1$

$\Rightarrow \lim_{x \to -1^+} f(x) = 0$

$\Rightarrow$ we need $\lim_{x \to -1^-} f(x) = 0$

$\Rightarrow \lim_{x \to -1^-} (2a - x) = 0$

$\Rightarrow 2a + 1 = 0$

$\Rightarrow \boxed{a = -\tfrac{1}{2},\ b = 1}$

(b) $g(x) = \begin{cases} x^2 - c^2, & x < 4 \\ cx + 20, & x \geq 4 \end{cases}$

Need $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x)$

$\Rightarrow \lim_{x \to 4^-} (x^2 - c^2) = \lim_{x \to 4^+} (cx + 20)$

$\Rightarrow 16 - c^2 = 4c + 20$

$\Rightarrow c^2 + 4c + 4 = 0$

$\Rightarrow (c + 2)^2 = 0$

$\boxed{c = -2}$

(c) $q(x) = \begin{cases} \frac{3\sin(0.12x)}{x}, & x \neq 0 \\ d, & x = 0 \end{cases}$

Need $\lim_{x \to 0} \frac{3\sin(0.12x)}{x} = d$

$\Rightarrow \lim_{x \to 0} 3(0.12) \frac{\sin(0.12x)}{0.12x} = d$

$\Rightarrow \boxed{d = 0.36}$

Bonus Problems: (You must complete all problems in the actual test to be eligible).

1. (1 point) Let $d(x)$ be a function that is differentiable at all real values of x. State the definition to compute its derivative

$$d'(x) = \lim_{h \to 0} \frac{d(x+h) - d(x)}{h} \quad \text{or} \quad d'(x) = \lim_{x \to a} \frac{d(x) - d(a)}{x - a}$$

2. (a) (2 points) Assume $f(x)$ is the function f as stated in problem 2. Using your answer to problem 2 part (v), find slope of the tangent line to f at the point where $x = 1$.

Slope $= f'(x)$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

from 2(v) $\rightarrow = \lim_{h \to 0} (2 - 2x - h) = 2 - 2x$

when $x = 1$,

Slope $= 2 - 2(1) = \boxed{0}$

(b) (2 points) Using your answer to part (a), find the equation of the tangent line to f when $x = 1$.

when $x = 1$, $y = 5$

$\Rightarrow$ tangent line: $\boxed{y - 5 = 0(x - 1)}$ or $\boxed{y = 5}$

3. (4 points) Use the definition of the derivative to find the derivative of $c(x) = \cos x$

$c'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \lim_{h \to 0} \left[\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right]$

$\boxed{= -\sin x}$

4. Suppose $f(x) = 3x^4 + 3x + 3\sin x - 2\cos x$. Find

(a) (2 points) $f'(x) = \boxed{12x^3 + 3 + 3\cos x + 2\sin x}$

(b) (1 point) The slope of the tangent line to $f(x)$ when $x = 0$.

$f'(0) = 0 + 3 + 3\cos 0 + 2\sin 0$

$\boxed{= 6}$