MATHEMATICS 391 **FINAL EXAM** **Spring, 2019**

NAME ____________________________

SECTION______________

INSTRUCTOR____________________________

Do all your work in this answer booklet. If you need extra space, use the facing pages.

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (7 points each):

(a) $\dfrac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}.$

(b) $(1+x^2)\,\dfrac{dy}{dx} - 2xy = x.$

(c) $$\frac{d^2y}{dt^2} + 9y = 81\sec^2(3t).$$

(2) Solve the following initial value problems (7 points each):

(a) $(e^x(\cos(xy) - y\sin(xy)))dx \ + \ (2y - e^x x\sin(xy))dy \ = \ 0 \qquad y(0) = 3$

(b) $y'' \ + \ 5y' \ = \ 25 + 100e^{5t}, \quad y(0) = 0, \ y'(0) = 0.$

(3) (11 points) A 500 gallon tank initially contains 100 gallons in which are dissolved 50 pounds of salt. The tank is flushed by pumping pure water into the tank at a rate of 3 gallons per minute and a well-mixed solution is pumped out at a rate of 1 gallon per minute. Compute the time when the tank has filled, then write the initial value problem which describes amount of salt in the tank at any time before the tank is full. Use the equation to compute the concentration of salt in the tank at the moment when it is filled.

(4) (12 points)For the differential equation:

$$(2 - x^4)y'' \ + \ (2x - 4)y' \ + \ (2x^2)y \ = \ 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first three nonzero terms of the solution with $y(0) = 12, y'(0) = 0$.

(5) (12 points)(a) Compute the sine series for the function f such that $f(x) = x(4-x)$ on the interval $[0, 4]$.

(b) Compute the solution $u(x,t)$ for the partial differential equation with x in the interval $[0, 4]$ and $t > 0$:

$$\begin{aligned} 3u_t &= u_{xx} \quad \text{with} \\ u(0,t) &= u(4,t) = 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)} \\ u(x,0) &= x(4-x) \quad \text{for } 0 < x < 4 \qquad \text{(initial conditions)} \end{aligned}$$

PartII Answer all sections of three (3) questions out of the five (5) questions in this part (10 points each).

(6) For the equation $t^2y'' - 5ty' + 9y = 0 \quad (t > 0)$, $y_1(t) = t^3$ is a solution.

(a) Use the method of Reduction of Order to obtain a second, independent solution.

(b) Solve the equation directly, using that it is an Euler Equation.

(c) Use the Wronskian of the pair of solutions to show that the pair form a fundamental set for the given differential equation.

(7) (a) State the definition of the Laplace transform and use it to compute the Laplace transform of the function f with $f(t) = t$.

(b) A hanging spring is stretched 18 inches ($= 1.5$ feet) by a mass with weight 60 pounds.

Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight starts from rest at the equilibrium position and is subjected to an external force of $5\cos(7t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

(8) (a) Compute the general solution of the differential equation

$$y^{(8)} + 8\,y^{(6)} + 16y^{(4)} = 0.$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients*. Do not attempt to determine the coefficients.

$$y^{(8)} + 8\,y^{(6)} + 16y^{(4)} = 2t + 5 - 8te^{-2t} - e^{t}\cos(2t) - 5\sin(2t).$$

(9) For the differential equation $(x^2-2x^3)y''+(6x)y'+(6+x)y=0$ show that the point $x=0$ is a regular singular point (either by using the limit definition or by computing the associated Euler equation). Compute the recursion formula for the series solution corresponding to the larger root of the indicial equation. With $a_0=4$, compute the first four nonzero terms of the series.

(10) Assume that y_1 and y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t.

(a) Show that $C_1 y_1 + C_2 y_2$ is a solution of the equation for any values of the constants C_1, C_2.

(b) Define the Wronskian of the pair y_1, y_2.

(c) Compute the Wronskian of $y_1(t) = t, y_2(t) = t^2$. Explain why the pair $\{t, t^2\}$ cannot be a pair of solutions defined on the whole real line of any second order, linear, homogeneous equation.