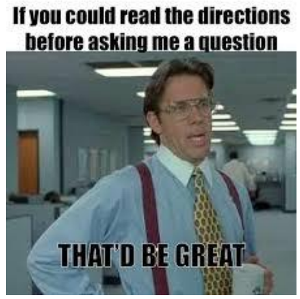
Math 346 Test 2
July 19, 2018

Name: ______________________________SOLUTIONS______________________________________

Note that both sides of each page may have printed material.



Instructions:

1. Read the instructions.

2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!

3. Complete all problems in the actual test. Fully justify! Bonus problems are optional, and will only be counted if all parts of all other problems are attempted.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.

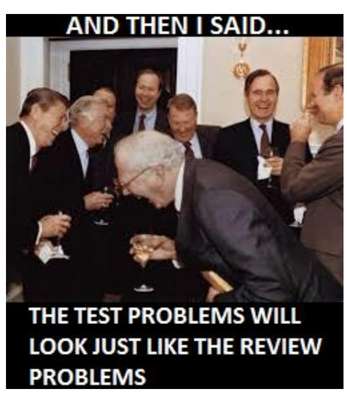
5. Write neatly so that I am able to follow your sequence of steps and box your answers.

6. Read through the exam and complete the problems that are easy (for you) first!

7. Scientific calculators are allowed but not required. **Graphing calculators are strictly forbidden!** You are also NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the internet, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**

8. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

9. Other than that, have fun and good luck!



1. Let $D: P_2 \to P_2$ be the differentiation operator $D(p) = p'(x)$.

(a) (10 points) Find $[D]_B$, where $B = \{2, 2-3x, 2-3x+8x^2\}$

$$[D]_B = \left[\left[D \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right]_B \middle| \left[D \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \right]_B \middle| \left[D \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix} \right]_B \right]$$

$$= \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_B \middle| \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}_B \middle| \begin{pmatrix} -3 \\ 16 \\ 0 \end{pmatrix}_B \right] \text{..(1)}$$

$$= \boxed{\begin{pmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{pmatrix}} \text{..(2)}$$

Note, we transformed polynomials to vectors using the isomorphism $a_0 + a_1 x + a_2 x^2 \mapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

Hence, $D(a_0 + a_1 x + a_2 x^2) = D\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = a_1 + 2a_2 x = \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix}$

To go from the matrix in line (1) to its equivalent in basis B, you have a few options. Two common options would be:

(i) Use the $\left(\begin{matrix} new \\ basis \end{matrix} \middle| \begin{matrix} old \\ basis \end{matrix} \right)$ method: Find the reduced row echelon form of $\left(\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & -3 & -3 \\ 0 & -3 & -3 & 0 & 0 & 16 \\ 0 & 0 & 8 & 0 & 0 & 0 \end{array} \right)$. The matrix in line (2) will appear on the right side.

(ii) Find $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ individually, via setting $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$ and solving for the c_i, either by solving the mini systems or by inspection.

(b) (5 points) Use part (a) to compute $[D(6-6x+24x^2)]_B$.

Use the equation $[D(\vec{p})]_B = [D]_B (\vec{p})_B$. This gives:

$$[D(6-6x+24x^2)]_B = [D]_B [6-6x+24x^2]_B$$
$$= \begin{pmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 13 \\ -16 \\ 0 \end{pmatrix}}$$

Note: We can find $[6-6x+24x^2]_B$ by either method (i) or (ii) shown in part (a).

(c) (5 points) Compute $D(6-6x+24x^2)$ using the above.

There are several ways to find this, but using our answer above means to use the coordinates. Since $[D(6-6x+24x^2)]_B =< 13, -16, 0 >$, we get

$$D(6-6x+24x^2) = 13\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - 16\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} -6 \\ 48 \\ 0 \end{pmatrix}}$$

It's easy to see this holds by computing derivatives directly.

2. (a) (5 points) Prove that $\boldsymbol{W = \{A \in M_{22} : A^T = A\}}$ is a subspace of $\boldsymbol{M_{22}}$.

Pf: We need to show W is (i) closed under addition and (ii) closed under scalar multiplication. Let $A, B \in W$. Then $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$ for $a, b, c, d, e \in \mathbb{R}$ and let $k \in \mathbb{R}$ be a scalar.

(i) <u>Closure under addition:</u> $(A+B)^T = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}^T = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix} = A + B$. This means $(A+B) \in W$, since it is its own transpose. Since $A, B \in W \Longrightarrow (A+B) \in W$, we have W is closed under addition.

(ii) <u>Closure under scalar multiplication:</u> $(kA)^T = \begin{pmatrix} ka & kb \\ kb & kc \end{pmatrix}^T = \begin{pmatrix} ka & kb \\ kb & kc \end{pmatrix} = kA$. This means $kA \in W$, since it is its own transpose. Since $A \in W \Longrightarrow kA \in W$, we have W is closed under scalar multiplication. ■

Alternatively, we could have used the properties of transposition:

Closure under addition: $(A+B)^T = A^T + B^T = A + B$, and
Closure under scalar multiplication: $(kA)^T = kA^T = kA$.

In each of the above equations, the first equal sign follows by general properties of the transposition, while the second equal sign follows from how the members of W are defined.

(b) (10 points) Let $\boldsymbol{A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$, $\boldsymbol{A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$ and $\boldsymbol{A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$. Prove that $\boldsymbol{B = \{A_1, A_2, A_3\}}$ is a basis for $\boldsymbol{W}$ defined above.

Pf: We need to show that the set $B = \{A_1, A_2, A_3\}$ is (i) linearly independent and (ii) spans W.

Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be any element in W and set $c_1A_1 + c_2A_2 + c_3A_3 = A$, where the c_i are scalars.

Then $c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. And so $c_1 = a, c_2 = c, c_3 = b$ will solve the equation. Thus A is a linear combination of the vectors in B, and so B spans W.

If we let $A = \vec{0} \in M_{22}$, then the above equation becomes,

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And we immediately obtain that $c_1 = c_2 = c_3 = 0$ is the only solution. Therefore, B is linearly independent.

■

3. **(a) (5 points) Prove that if $\boldsymbol{T_1: U \to V}$ and $\boldsymbol{T_2: V \to W}$ are linear transformations of vector spaces, then the composition $\boldsymbol{T_2 \circ T_1: U \to W}$ is also a linear transformation.**

Pf: We need to show that $T_2 \circ T_1$ has the addition property and the homogeneity property.

Let $\overrightarrow{u_1}, \overrightarrow{u_2} \in U$, and let k be any scalar in our field.

<u>$T_2 \circ T_1$ is additive:</u> Note that

$T_2 \circ T_1(\overrightarrow{u_1} + \overrightarrow{u_2}) = T_2(T_1(\overrightarrow{u_1}, +\overrightarrow{u_2}))$ ……………………………………..by definition of composition
$= T_2(T_1(\overrightarrow{u_1}) + T_1(\overrightarrow{u_2}))$ ……………………………….since T_1 is additive
$= T_2\big(T_1(\overrightarrow{u_1})\big) + T_2\big(T_1(\overrightarrow{u_2})\big)$ ………………………..since T_2 is additive
$= T_2 \circ T_1(\overrightarrow{u_1}) + T_2 \circ T_1(\overrightarrow{u_2})$ ………………………..by definition of composition

Thus, $T_2 \circ T_1$ is additive.

<u>$T_2 \circ T_1$ is homogeneic:</u> Note that
$T_2 \circ T_1(k\overrightarrow{u_1}) = T_2\big(T_1(k\overrightarrow{u_1})\big)$ ……………………………………………….by definition of composition
$= T_2\big(kT_1(\overrightarrow{u_1})\big)$ ……………………………………………….by homogeneity property of T_1
$= kT_2\big(T_1(\overrightarrow{u_1})\big)$ ……………………………………………….by homogeneity property of T_2
$= kT_2 \circ T_1(\overrightarrow{u_1})$ ……………………………………………….by definition of composition

Thus $T_2 \circ T_1$ possesses the homogeneity property.

Since $T_2 \circ T_1$ has both these properties, it is a linear transformation. ■

(b) (5 points) A linear transformation is called an *isomorphism* if it is one-to-one and onto. If there is an isomorphism between two vector spaces, then the vector spaces are said to be *isomorphic*. Prove that the property of being isomorphic is *transitive*. That is, prove that if $\boldsymbol{U, V}$ and $\boldsymbol{W}$ are vector spaces, and $\boldsymbol{U}$ is isomorphic to $\boldsymbol{V}$ and $\boldsymbol{V}$ is isomorphic to $\boldsymbol{W}$, then $\boldsymbol{U}$ is isomorphic to $\boldsymbol{W}$.

Pf: Assume that U and V are isomorphic, and V and W are isomorphic. That is, there exist bijective linear transformations $T_1: U \to V$ and $T_2: V \to W$. We show that U and W are isomorphic. That is, there exists a bijective linear transformation from U to W.

We claim that $T_2 \circ T_1: U \to W$ is such a transformation. By part (a) above, it is a linear transformation. And we've previously shown (in class) that a composition is both one-to-one and onto if the functions that make up the composition are each one-to-one and onto. So, the composition of bijections is itself a bijection.

Hence, $T_2 \circ T_1$ is a bijective linear transformation from U to W, and so we have U is isomorphic to W.

■

4. **Let** $A = \begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 2 & 4 & 7 & 7 & 0 \\ 3 & 6 & 9 & 9 & -1 \\ 1 & 2 & 4 & 4 & 1 \end{pmatrix}$

(a) (8 points) Find a basis for the column space of A

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \\ R2-2R1 \\ 3R1-R3 \\ R4-R1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ R4-R2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R1-3R2 \\ \\ \\ R3-R4 \end{matrix}$$

$$\Rightarrow \boxed{Basis\ for\ col(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

(b) (6 points) Find a basis for the null space of A

By the above, if we consider $A\vec{x} = \vec{0}$, we get $x_2 = t$, $x_4 = r$. And consequently, $R3$ gives $x_5 = 0$, $R2$ gives $x_3 = -x_4 = -r$, and $R1$ gives $x_1 = -2x_2 = -2t$.

So, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ -r \\ r \\ 0 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ solves the homogeneous equation.

And we get $\boxed{Basis\ for\ Null(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}}$

(c) (4 points) Find $\boldsymbol{rank(A)}$ and $\boldsymbol{Nullity(A)}$.

$\boxed{Rank(A) = 3} = \#\ of\ pivots\ in\ RREF(A)$

$\boxed{Nullity(A) = 2} = \#\ of\ parameters\ in\ (b) = \dim(Null(A)) = (\#\ of\ columns) - Rank(A)$

(d) (2 points) The correct answer to (c) is an example of a general principle. What is this principle?

The Dimension Theorem (for Matrices)

5. (a) (5 points) Let S be a finite set of linearly independent vectors. Prove that any non-empty subset of S is also a linearly independent set. Hint: a proof by contradiction is a good way to go.

Pf: Let $B \subseteq S$ and $B \neq \emptyset$. Clearly, if $B = S$ the claim is true. So let us assume $B \neq S$.

For convenience, let us write $S = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}, \dots, \overrightarrow{b_m}, \overrightarrow{b_{m+1}}, \dots, \overrightarrow{b_n}\}$ and let B be the subset containing the first m elements, that is, $B = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}, \dots, \overrightarrow{b_m}\}$.

Assume, to the contrary that B is linearly dependent. Then the equation,

$$c_1\overrightarrow{b_1} + c_2\overrightarrow{b_2} + \cdots + c_m\overrightarrow{b_m} = \vec{0}$$

has a non-trivial solution for the c_i. That means, at least one of the $c_i \neq 0$. WLOG, assume $c_1 \neq 0$.

Then the equation $c_1\overrightarrow{b_1} + 0 \cdot \overrightarrow{b_2} + \cdots + 0 \cdot \overrightarrow{b_m} + 0 \cdot \overrightarrow{b_{m+1}} + \cdots + 0 \cdot \overrightarrow{b_n} = \vec{0}$ shows that there is a $\vec{0}$-linear combination of vectors in S where the scalars are not all zero. This yields that S is dependent, and we obtain a contradiction.

■

(b) (5 points) Suppose $\boldsymbol{S} = \{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}\}$ is a linearly independent set. Prove that $\boldsymbol{B} = \{\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}, \vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}\}$ is also linearly independent. $\vec{u} + \vec{v}$

Pf: Set $c_1(\vec{u} + \vec{v}) + c_2(\vec{u} + \vec{v}) = \vec{0}$. We need to show that $c_1 = c_2 = 0$. Rewriting the left side of the first equation, we get $(c_1 + c_2)\vec{u} + (c_1 - c_2)\vec{v} = \vec{0}$. Since $\vec{u}$ and $\vec{v}$ are linearly independent and $(c_1 + c_2)$ and $(c_1 - c_2)$ are scalars, we must have

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0$$

Solving this system gives $c_1 = c_2 = 0$, and hence B is linearly independent.

■

6. **(a) (10 points) Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation obtained by first reflecting over the x-axis, then rotating counter-clockwise by an angle of $\frac{\pi}{2}$. Find the standard matrix for T. Is T one-to-one? Justify.**

Using the transformations geometrically as shown in class or based on the handout posted on the class webpage: http://math.sci.ccny.cuny.edu/docs?name=MatrixTransformsInR2andR3.pdf
We have the following transformations:

Set $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to$ the matrix transformation to reflect over the x-axis.

Set $T_2 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to$ the matrix transformation to rotate CCW by $\theta = \frac{\pi}{2}$.

Then we can construct $[T]$ by first applying T_1 then applying T_2. That is,

$$[T] = T_2 \circ T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Since $\det[T] \neq 0$, T is 1-1 by the equivalence theorem.

(b) (10 points) Determine if the following linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ is one-to-one, and whether T^{-1} exists. T is defined via

$$\begin{aligned} w_1 &= 2x_1 + 2x_2 + x_3 \\ w_2 &= 2x_1 + x_2 - x_3 \\ w_3 &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

Note that $[T] = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$ and so $\det[T] = -3 \neq 0$.

Since $\det[T] \neq 0$, we have that T is one-to-one.

Since T is an operator (and hence the dimension of its domain space and range space are the same), T is also onto. Since T is one-to-one and onto, it is invertible, and so $[T]^{-1}$ exists.

All these statements follow via the equivalence theorem.

Bonus Problems: You must attempt all parts of all other problems to be eligible.

1. Consider the matrix $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$.

(a) (10 points) Find the eigenvalues and corresponding eigenvectors of A.

Set $\det\begin{pmatrix} \lambda - 3 & -4 \\ 1 & \lambda + 2 \end{pmatrix} = 0$

$\Rightarrow (\lambda - 3)(\lambda + 2) + 4 = 0$
$\Rightarrow \lambda^2 - \lambda - 2 = 0$
$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$
$\Rightarrow \boxed{\lambda_1 = 2, \lambda_2 = -1} \rightarrow eigenvalues$

If $\lambda_1 = 2$, we get the system

$$\left(\begin{array}{cc|c} -1 & -4 & 0 \\ 1 & 4 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R2 \\ R1 + R2 \end{array}$$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4t \\ t \end{pmatrix} = t\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$
$\Rightarrow \boxed{\overrightarrow{\lambda_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}} \rightarrow$ eigenvector for $\lambda_1 = 2$

If $\lambda_2 = -1$, we get the system

$$\left(\begin{array}{cc|c} -4 & -4 & 0 \\ 1 & 1 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R2 \\ 4R2 + R1 \end{array}$$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$\Rightarrow \boxed{\overrightarrow{\lambda_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}} \rightarrow$ eigenvector for $\lambda_2 = -1$

(b) (2 points) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Set $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, P = \left(\overrightarrow{\lambda_1} \middle| \overrightarrow{\lambda_2}\right)$

$\Rightarrow \boxed{D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, P = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}}$

Note that $P^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$, and so, one can check that

$A = PDP^{-1} = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$

(c) (5 points) Compute A^5, write as a 2x2 matrix.

$$A^5 = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^5\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 43 & 44 \\ -11 & -12 \end{pmatrix}}$$

(d) (3 points) Solve the following system for the functions $y_1(t)$ and $y_2(t)$, subject to the initial conditions $y_1(0) = 1$ and $y_2(0) = 2$.

$$\begin{Bmatrix} y_1{}'(t) & = & 3y_1(t) + 4y_2(t) \\ y_2{}'(t) & = & -y_1(t) - 2y_2(t) \end{Bmatrix}$$

From part (a) we have $\lambda_1 = 2,\ \vec{\lambda_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1,\ \vec{\lambda_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\Longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$ → if you get up to here, you'll get 2 out of the 3 points

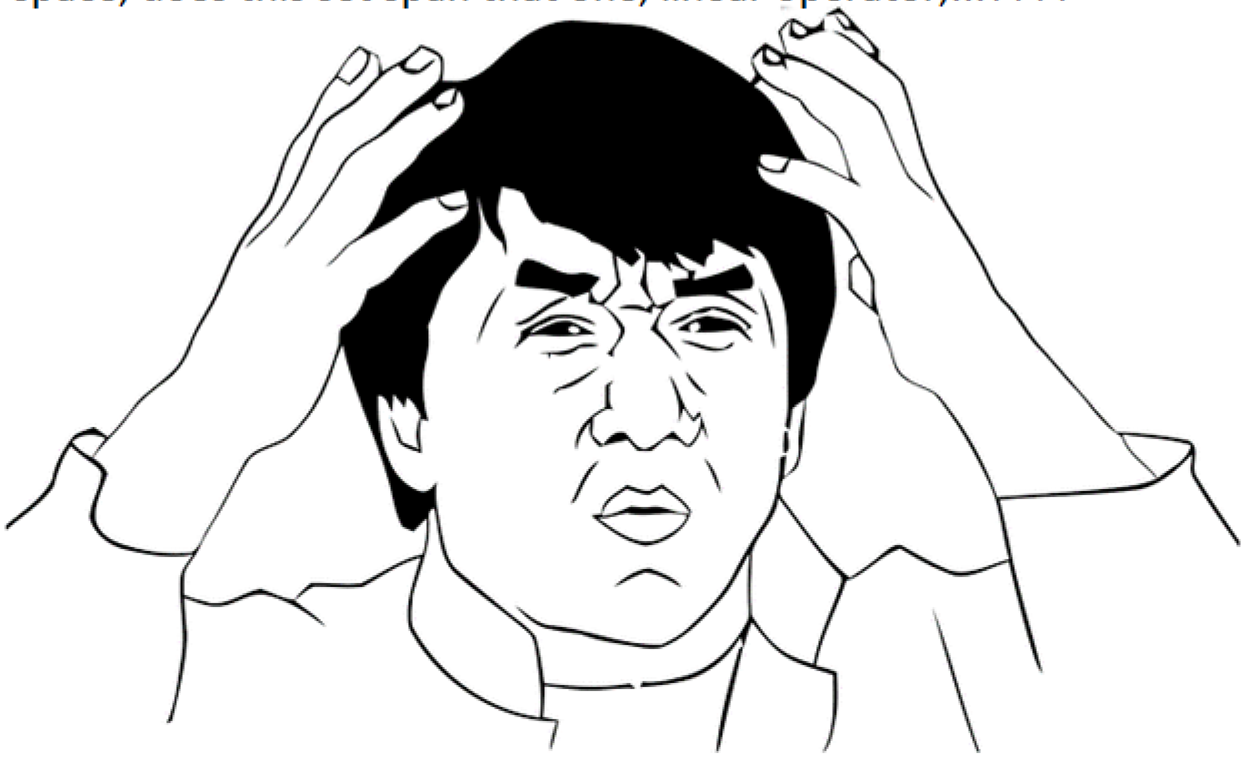
This means,
$y_1 = -4c_1 e^{2t} - c_2 e^{-t}$
$y_2 = c_1 e^{2t} + c_2 e^{-t}$

Then, applying the initial conditions to each of these equations,
$y_1(0) = 1 \Longrightarrow 1 = -4c_1 - c_2$
$y_2(0) = 2 \Longrightarrow 2 = c_1 + c_2$

Solving this system gives $c_1 = -1$ and $c_2 = 3$. Thus,

$\Longrightarrow \boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -\begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{2t} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}}$ OR $\boxed{\begin{matrix} y_1 = 4e^{2t} - 3e^{-t} \\ y_2 = -e^{2t} + 3e^{-t} \end{matrix}}$

?? Linear independence, span, basis, null space, subspace, vector space, does this set span that one, linear operator,...????



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