Math 346 Test 1

June 28, 2018

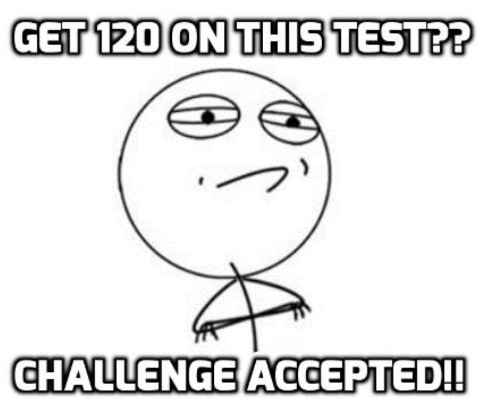
Name: ________________________________SOLUTIONS_______________________________________

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are allowed, but not required. **Graphing calculators are strictly forbidden!** You are also NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

May the force be with you. But you can't ask it to help you with your test.



1. Let $A = \begin{pmatrix} 2 & 2 & -2 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

(a) (15 points) Find A^{-1}

The row reduction method is, of course, valid to use here. However, the details may vary from student to student, so we will find A^{-1} using the adjoint method.

First, note $\det A = \det \begin{pmatrix} 1 & 0 & -3 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}^{R1-R2} = -2\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} = 2$, expanding along $C2$.

Now, $$Adj(A) = \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -4 & 3 & -2 \\ -6 & 4 & -2 \\ 4 & -2 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} -4 & -6 & 4 \\ 3 & 4 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

Then we have $A^{-1} = \frac{1}{\det A} Adj(A) = \boxed{\frac{1}{2}\begin{pmatrix} -4 & -6 & 4 \\ 3 & 4 & -2 \\ -2 & -2 & 2 \end{pmatrix}}$.

(b) (5 points) Use A^{-1} to solve the system $\begin{cases} \mathbf{2x} & + & \mathbf{2y} & - & \mathbf{2z} & = & \mathbf{1} \\ \mathbf{-x} & & & + & \mathbf{2z} & = & \mathbf{0} \\ \mathbf{x} & + & \mathbf{2y} & + & \mathbf{z} & = & \mathbf{1} \end{cases}$

If $A\vec{x} = \vec{b}$ and A^{-1} exists, then $\vec{x} = A^{-1}\vec{b}$

$$\Rightarrow \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & -6 & 4 \\ 3 & 4 & -2 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}}$$

2. Let $A = \begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$

(a) (12 points) Find $\det A$

Expanding along the 2[nd] column, we get $\det A = -1 \begin{vmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix}$

$$= - \begin{vmatrix} 3 & -3 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{vmatrix} R2 - R3$$

$$= (-1)(1) \begin{vmatrix} 3 & -3 \\ 2 & -1 \end{vmatrix} \textit{ along 3rd column}$$

$$= (-1)(1)(-3+6)$$

$$\Longrightarrow \boxed{\det A = -3}$$

(b) (2 points) Find $\det(A^{-3})$

$$\det(A^{-3}) = (\det A)^{-3} = (-3)^{-3} = \boxed{-\frac{1}{27}}$$

(c) (2 points) Find $\det(A^{-1})$

$$\det(A^{-1}) = (\det A)^{-1} = (-3)^{-1} = \boxed{-\frac{1}{3}}$$

(d) (4 points) Find $\det(2A^3A^{-1}A^TA^{-2})$

$$\begin{aligned} \det(2A^3A^{-1}A^TA^{-2}) &= 2^4 \cdot (\det A)^3 \cdot \frac{1}{\det A} \cdot \det A \cdot (\det A)^{-2} \\ &= 2^4 \cdot (\det A) \\ &= \boxed{-48} \end{aligned}$$

3. (a) (15 points) Solve the system by a method of your choice: $\begin{cases} 3x + 2y - z = 0 \\ x + y = -1 \\ x - z = 2 \end{cases}$

Write your solution in vector form.

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & -3 \end{array}\right) \begin{array}{c} R3 \\ R2 - R3 \\ 3R2 - R1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} R1 \\ R2 \\ R2 - R3 \end{array}$$

Set $z = t$, since the third column has no pivot.
From R2: $y + t = -3 \Longrightarrow y = -t - 3$
From R1: $x - t = 2 \Longrightarrow x = t + 2$

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t + 2 \\ -t - 3 \\ t \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

(b) (5 points) Could the above system be solved using Cramer's rule? Explain.

No. Because the system does not have a unique solution. (One could also say something like, "No, because the determinant of the coefficient matrix is zero.", but this, and similar claims would have to be verified. So, for example, you can't talk about what the determinant is if you didn't find the determinant or argued why it has to be a certain value. Something like "Since the RREF of the coefficient matrix has a row of zeros, its determinant must be zero; therefore, Cramer's rule cannot find the solution." also works.)

4. (a) <u>Prove or disprove</u> – Assuming the matrices shown have dimensions so that the operations are defined. (2 points each):

(i) **If A and B are invertible, then so is AB.** **This is TRUE!**

Pf: Assume $A_{n\times n}$ and $B_{n\times n}$ are invertible. Then A^{-1} and B^{-1}, and so does the product $B^{-1}A^{-1}$. Then $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$. This means that the inverse of AB exists, in fact, $(AB)^{-1} = B^{-1}A^{-1}$. ■

Alternatively,

Pf: Assume $A_{n\times n}$ and $B_{n\times n}$ are invertible. Then, by the equivalence theorem, $\det A \neq 0$ and $\det B \neq 0$. This means that $\det(AB) = \det A \cdot \det B \neq 0$. Then, again by the equivalence theorem, AB is invertible. ■

(ii) **If A and B are invertible, then so is $A - B$.** **This is FALSE!**

There are many ways to construct counter-examples here. Here is one way: Let A be any invertible matrix. Set $B = A$, then B is also invertible. Then, $A - B = \vec{0}$, which is not invertible. It is OK to use specific matrices as examples here.

(iii) **If $A^TA = A$, then A^T must be I_n.** **This is FALSE!**

Counter-example: Choose $A_{n\times n} = \vec{0}$. Then $A^T = \vec{0}$, so $A^TA = \vec{0} = A$, but clearly, $A^T \neq I_n$.

(iv) **If A is invertible, then $\det(A^{-1}BA) = \det(B)$.** **This is TRUE!**

Pf: By our determinant theorems,

$\det(A^{-1}BA) = \det A^{-1} \cdot \det B \cdot \det A = \frac{1}{\det A} \cdot \det B \cdot \det A = \det B$. ■

WARNING: If you wrote something similar to $det\left(A^{-1}BA\right) = det\,(I_nB) = det\,B$, you will not get credit. For this is tantamount to assuming the matrix multiplication is commutative. $A^{-1}BA \neq B$ in general. You canNOT choose what order you want to multiply here! Matrix multiplication is NOT commutative!

(v) **If B is a square matrix, then BB^T is symmetric.** **This is TRUE!**

Pf: Recall that A is symmetric iff $A^T = A$. Note that $(BB^T)^T = (B^T)^T(B)^T = BB^T$. Therefore, BB^T is symmetric, since it is equal to its transpose. ■

(vi) **If B is a square matrix, then $B + B^T$ is symmetric.** **This is TRUE!**

Pf: Since $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$, we have that $B + B^T$ is equal to its transpose. Hence it is symmetric. ■

(vii) **If $A^TA = A$, then A is symmetric, and $A = A^2$.** **This is TRUE!**

Pf: Assume that $A^TA = A$. Then $(A)^T = (A^TA)^T = A^T(A^T)^T = A^TA = A$, where the first and last equalities follow from our assumption. Since this means $A^T = A$, we have that A is symmetric. Moreover, since $A^T = A$, we have that $A^TA = AA = A^2 = A$. ■

5. (a) (8 points) Prove that a square matrix A is invertible if and only if $\det A \neq 0$.

There are many ways to go about this, here's one.

Pf: ($\Longrightarrow$): Assume $A_{n\times n}$ is invertible. Then any system $A\vec{x} = \vec{b}$ must be consistent, for $\vec{x}$ and $\vec{b}$ column vectors. To see this, simply multiply both sides on the left by A^{-1}, which exists since A is invertible, to obtain the unique solution $\vec{x} = A^{-1}\vec{b}$. But this would mean that, had we augmented the system and found the reduced row echelon form, the $RREF(A) = I_n$. In particular, this means that $\det(RREF(A)) = 1 \neq 0$. But since row operations only scale the determinant by non-zero scalars, this can only happen if $\det A \neq 0$.

($\Longleftarrow$): For the converse, assume A is not invertible. Then $RREF(A) \neq I_n$, by the equivalence theorem. But that means that $RREF(A)$ has a row of zeros, by a theorem in class. This means $\det(RREF(A)) = 0$. Since row operations only scale the determinant by some non-zero constant, we get that $\det A = 0$. Therefore, the contrapositive of the converse holds.

$\blacksquare$

Again, technically what you were asked to prove comes straight from the equivalence theorem, but you were required to make some sort of argument. You were allowed to use the equivalence theorem, as I did above, as long as you passed through some other statement in the theorem and connected the ideas. Knowing the theorem by rote is important, but it is also important to understand the logic that connects each of the statements together.

(b) (8 points) Prove that a square matrix A is invertible if and only if $A^T A$ is invertible.

Pf: For the forward implication, assume A is invertible. Then, by the above result, $\det A \neq 0$. But $\det A = \det A^T$, so that $\det A^T \neq 0$. But this means, $\det(A^T A) = \det A^T \cdot \det A \neq 0$. Again, by the above result, this means $A^T A$ is invertible.

For the converse, assume A is not invertible. Then part (a) gives us $\det A = 0$. We then have that $\det(A^T A) = \det A^T \cdot 0 = 0$. Since $\det(A^T A) = 0$, we have that $A^T A$ is not invertible, again by applying part (a).

$\blacksquare$

6. (10 points) Consider the system in problem 1(b), that is, the system

$$\begin{cases} \mathbf{2x} & + & \mathbf{2y} & - & \mathbf{2z} & = & \mathbf{1} \\ \mathbf{-x} & & & + & \mathbf{2z} & = & \mathbf{0} \\ \mathbf{x} & + & \mathbf{2y} & + & \mathbf{z} & = & \mathbf{1} \end{cases}$$

Use Cramer's Rule to solve for y only. (Do NOT solve for x or z!) No credit will be given for any other method.

Here, let A be the coefficient matrix of the system.

Set $D = \det A = 2$ (found in problem 1).

Then $D_y = \begin{vmatrix} 2 & 1 & -2 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -1\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ -1 & 2 \end{vmatrix} = (-1)(-3) - (1)(2) = 1$

This, by Cramer's rule: $y = \frac{D_y}{D} = \boxed{\frac{1}{2}}$

(And this is vindicated by our solution to problem 1(b))

Bonus Problems:

Definition: Vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots\ , \overrightarrow{v_n}$ are said to be linearly independent if the ONLY solution to the equation

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n} = \vec{0}$$

is the trivial solution, $c_1 = c_2 = \cdots = c_n = 0$.

Definition: A set B of vectors form a basis for a vector space if the set of vectors is (1) linearly independent and (2) they span the vector space—that is, every vector in the vector space can be expressed as a linear combination of vectors in B.

1. **(5 points) Show that the functions x^2 and x^3 are linearly independent.**

   Pf: Set $c_1x^2 + c_2x^3 = \vec{0} = 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$. By equating coefficents, we find that $c_1 = c_2 = 0$.

   ■

2. **(5 points) Show that the set $B = \{\vec{\imath}, \vec{\jmath}, \vec{k}\} = \{< 1, 0, 0 >, < 0, 1, 0 >, < 0, 0, 1 >\}$ is a basis for $\mathbb{R}^3$**

   Pf: Let $< a, b, c >$ be any vector in $\mathbb{R}^3$. Since $< a, b, c > = a < 1,0,0 > +b < 0,1,0 > +c < 0,0,1 >$, we have that B spans $\mathbb{R}^3$, since we can express any vector as a linear combination of vectors in B.

   Secondly, setting $c_1 < 1,0,0 > +c_2 < 0,1,0 > +c_3 < 0,0,1 > = \vec{0} =< 0,0,0 >$, we get that $< c_1, c_2, c_3 > = < 0,0,0 >$. By equating corresponding components, we have $c_1 = c_2 = c_3 = 0$. Thus, the elements of B are linearly independent.

   Since the elements of B are linearly independent and span $\mathbb{R}^3$, the set B is a basis of $\mathbb{R}^3$. ■

3. **(5 points) Prove that, in a vector space, $(-1)\vec{u} = -\vec{u}$.**

   Pf: By the problem below, we know that $0\vec{u} = \vec{0}$. Thus we have,

   $$\begin{aligned} \vec{0} &= 0\,\vec{u} \\ &= (1 + (-1))\vec{u} \text{ by properties of the real numbers} \\ &= 1\vec{u} + (-1)\vec{u} \text{ by the distributive law axiom} \\ &= \vec{u} + (-1)\vec{u} \text{ by axiom 10} \end{aligned}$$

   Since adding $(-1)\vec{u}$ to the vector $\vec{u}$ results in $\vec{0}$, $(-1)\vec{u}$ is the negative of $\vec{u}$, by our vector space axioms. Thus, we may denote it by $(-1)\vec{u} = -\vec{u}$. ■

4. (5 points) Prove that, in a vector space, $0\vec{u} = \vec{0}$

   Pf: Note that $0\vec{u} = (0 + 0)\vec{u}$ by the properties of $0 \in \mathbb{R}$
   $= 0\vec{u} + 0\vec{u}$ by the distributive law axiom

   Since $0\vec{u}$ is not changed when $0\vec{u}$ is added to it, $0\vec{u}$ satisfies the property of the $\vec{0}$ vector. Since the zero vector in a vector space is unique, we must have $0\vec{u} = \vec{0}$. ■

GOT 120?