Math 308 Test 2
July 24, 2018

Name: ____

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.

2. Everything you write must be fully justified.

3. Complete all problems in the actual test. Bonus problems are, of course, optional. Bonus problems will only be counted if all parts of all other problems are attempted.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (that's not really possible in this test anyway…)

5. Write neatly so that I am able to follow your sequence of steps.

6. Read through the exam and complete the problems that are easy (for you) first!

7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.

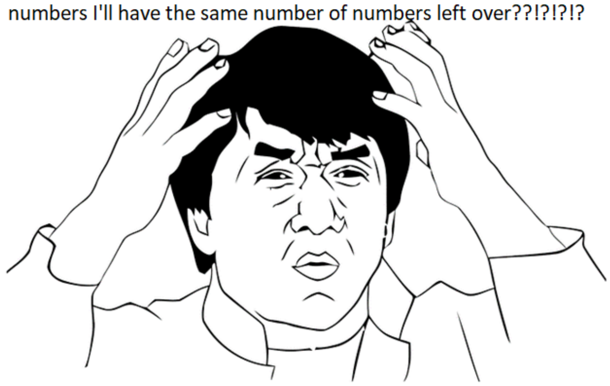
8. In fact, **cell phones should be out of sight!**

9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

10. **Remember: if you mess up on a definition in a problem, you will get a zero for that problem.** Use the definitions from class. If you want to use another, you must first prove it is equivalent to the class' definition.

11. Other than that, have fun and good luck!

So, there are infinitely many real numbers and infinitely many integers, yet, somehow, there are more real numbers than there are integers. And if I throw away all the integers from the real numbers I'll have the same number of numbers left over??!?!?!?



SET THEORY:
Making people go crazy. Since 1891.
FML!

1. (a) (5 points) Prove $\mathbb{R}$ is uncountable. Fully justify.

(b) (5 points) Prove that $\left|\mathbb{N}^{\mathbb{N}}\right| = |\mathbb{R}|$

2. (20 points) Suppose A is a non-empty set of real numbers that has a lower bound. Prove that A has an infimum, that is, a greatest lower bound. Hint: You may use the completeness axiom for $\mathbb{R}$.

3. (a) (15 points) Let A and B be non-empty sets such that $|A| = |B|$. Prove that, if C is a non-mepty set, then $|C^A| = |C^B|$.

(b) (5 points) Assume now that $|A| \leq |B|$, prove that $|C^A| \leq |C^B|$.

4. (20 points) Prove that if $f: A \to B$ is surjective, then there exists $g: B \to A$ such that g is injective and $g \circ f = id_A$.

5. (20 points) Let A be a set of real numbers. We say that A is *dense* (in $\mathbb{R}$) if for any two $x, y \in \mathbb{R}$ with $x < y$ there exists an $a \in A$ such that $x < a < y$. Assume A is countable. Prove that $\mathbb{R} \backslash A$ is dense.

   Hint: You may use contradiction here. Assuming the contrary implies that the interval $(x, y) \subseteq A$. Explain why this must be the case under this assumption and show how this leads to a contradiction.

6. (a) (5 points) Prove that $\mathbb{Q}$ is countable. From scratch!

(b) (5 points) Prove that for any set A, $|A| < |\mathcal{P}(A)|$.

Bonus Problems: (You must attempt all other parts of all other problems to be eligible. If with the bonus points, your grade exceeds 100, I'll add the excess to your first test.)

1. (10 points) (a) Let $s: \mathbb{N} \to \mathbb{R}$ be a non-decreasing sequence, that is, $s_1 \leq s_2 \leq s_3 \leq \cdots$ Suppose that the sequence is bounded above. Prove that the sequence s_n is convergent.
   Hint: It suffices to show that if $\{s_1, s_2, s_3, \dots\}$ is an ordered, infinite set of real numbers and is bounded above by L, then $\lim_{n\to\infty} s_n = L$. This is a proofs class, so you must show a limit exists using a rigorous definition. See glossary on last page.

   (b) (5 points) The above is actually a well-known theorem. What's the name of the theorem?

2. (15 points) Prove that, given any real number x, there exists a sequence $s: \mathbb{N} \to \mathbb{Q}$ with $s_n \to x$. (See glossary.)

   For example, choose $x = \pi$, then the sequence $s = \{(1,3), (2,3.1), (3,3.14), (4,3.141), (5,3.1415), \dots\}$ approaches π.

   Hint: Use the denseness of $\mathbb{Q}$ theorem. By this theorem, there must exist rational numbers $s_1, s_2, s_3, \dots, s_n, \dots$ in the intervals $(x-1, x+1), \left(x - \frac{1}{2}, x + \frac{1}{2}\right), \left(x - \frac{1}{3}, x + \frac{1}{3}\right), \dots, \left(x - \frac{1}{n}, x + \frac{1}{n}\right), \dots$ Show that such a sequence approaches x.

Glossary:

Definition: A (real) *sequence* is a function s from $\mathbb{N} \to \mathbb{R}$.

We denote the outputs of a sequence $s(1), s(2), s(3), \ldots$ by $s_1, s_2, s_3, \ldots$
Sometimes we define or refer to a sequence by an ordered list $\{s_1, s_2, s_3, \ldots\}$ of its outputs.
We may also denote an arbitrary term of a sequence by s_n or the entire sequence by $\{s_n\}$.

Definition: Let $L \in \mathbb{R}$. A sequence s is said to *converge to* L or *approach* L if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that, whenever $n > N$ we have $|s_n - L| < \epsilon$.

If a sequence s converges to L we write $\lim_{n \to \infty} s_n = L$ or $\lim s_n = L$ or $s_n \to L$.

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Me after reading through take home