Math 308

Examples to clarify solution to test 2 problem 3(b).

Note that the notation chosen here reflects the notation used in the solution file, so this should be read alongside the solution.

As a concrete example, consider the following sets: $A = \{1,2\},\ B = \{a,b,c\},\ C = \{x,y\}$; so $|A| \leq |B|$.

Let $g: A \to B$, $f: A \to C$, and $\bar{f}: B \to C$. And let $\varphi: A \to B$ be an injection. We wish to create an injective map $F: C^A \to C^B$, this would show $|C^A| \leq |C^B|$. We have the following:

The elements of B^A	The elements of C^A	The elements of C^B
$g_1 = \{(1,a),(2,a)\}$	$f_1 = \{(1,x),(2,x)\}$	$\bar{f}_1 = \{(a,x),(b,x),(c,x)\}$
$g_2 = \{(1,a),(2,b)\}$	$f_2 = \{(1,x),(2,y)\}$	$\bar{f}_2 = \{(a,x),(b,x),(c,y)\}$
$g_3 = \{(1,a),(2,c)\}$	$f_3 = \{(1,y),(2,x)\}$	$\bar{f}_3 = \{(a,x),(b,y),(c,x)\}$
$g_4 = \{(1,b),(2,a)\}$	$f_4 = \{(1,y),(2,y)\}$	$\bar{f}_4 = \{(a,x),(b,y),(c,y)\}$
$g_5 = \{(1,b),(2,b)\}$		$\bar{f}_5 = \{(a,y),(b,x),(c,x)\}$
$g_6 = \{(1,b),(2,c)\}$		$\bar{f}_6 = \{(a,y),(b,x),(c,y)\}$
$g_7 = \{(1,c),(2,a)\}$		$\bar{f}_7 = \{(a,y),(b,y),(c,x)\}$
$g_8 = \{(1,c)\ \ (2,b)\}$		$\bar{f}_8 = \{(a,y),(b,y),(c,y)\}$
$g_9 = \{(1,c),(2,c)\}$		

Now, remember the goal is to show that: given an injection φ from $A \to B$, we will be able to create a 1-1 function $F: C^A \to C^B$. That is, based on the φ chosen, we want to induce a 1-1 function from the elements in the second column to the elements in the third column. Note there are 6 possible injections that could be φ in this example: $g_2, g_3, g_4, g_6, g_7,$ and g_8. We will show how things will break down for three of them.

<u>Example 1:</u> Choose $\varphi = g_2$, and choose $p = x$.

Then $B' = \{a,b\}$. And since $c \notin B'$, F will only map to functions in the third column above such that $\bar{f}(c) = p = x$ and at the same time, $\bar{f}(n) = f(\varphi^{-1}(n))$ if $n \in B'$. This means that the first two outputs of the functions in the third column will mimic the outputs of the functions in the second column based on how φ assigns elements in B to elements in A, while the third output of the functions in the third column, all map to x which is our chosen p-value. This results in the following injection:

$$f_1 \mapsto \bar{f}_1$$

$$f_2 \mapsto \bar{f}_3$$

$$f_3 \mapsto \bar{f}_5$$

$$f_4 \mapsto \bar{f}_7$$

Think you got it? Go to the top half of the next page for another example, which we will explain slightly differently, yet equivalently. Do not scroll to the third page yet.

Example 2: Choose $\varphi = g_6$, and choose $p = y$.

Then $B' = \{b, c\}$. Since $a \notin B'$, we want to choose functions in the third column so that $\bar{f}(a) = y$. Moreover, we want to think of our b as the 1 in column 2, and our c as the 2 in column 2, since $\varphi^{-1}(b) = 1$ and $\varphi^{-1}(c) = 2$—if we assume $\varphi = g_6$. Thus our injection is given by $F: C^A \to C^B$ described by:

$$f_1 \mapsto \overline{f_5}$$

$$f_2 \mapsto \overline{f_6}$$

$$f_3 \mapsto \overline{f_7}$$

$$f_4 \mapsto \overline{f_8}$$

So you see, the above happens without loss of generality. If we choose any of the injective functions in the first column, and any member of C to be our p-value, we automatically induce the set B' as well as an injection F from C^A to C^B.

Try this one on your own (there are other scenarios using these sets A, B, and C, you can construct even more examples if you want). I will give you the choices. DO NOT SCROLL TO THE NEXT PAGE!

Example 3: Choose $\varphi = g_7$, and choose $p = x$

State what B' is, and then describe the injection that results as was done in the previous two examples. When you think you have the answer, scroll to the next page to check if you're correct.

Example 3: Choose $\varphi = g_7$, and choose $p = x$.

Then $B' = \{a, c\}$. So we want to choose functions in the third column so that $\bar{f}(b) = x$ and our c behaves like the "1" in column 2, and our a behaves like the "2" in column 2. Thus our injection is given by $F: C^A \to C^B$ described by:

$$f_1 \mapsto \overline{f_1}$$

$$f_2 \mapsto \overline{f_5}$$

$$f_3 \mapsto \overline{f_2}$$

$$f_4 \mapsto \overline{f_6}$$