Math 308 Test 2
July 24, 2018

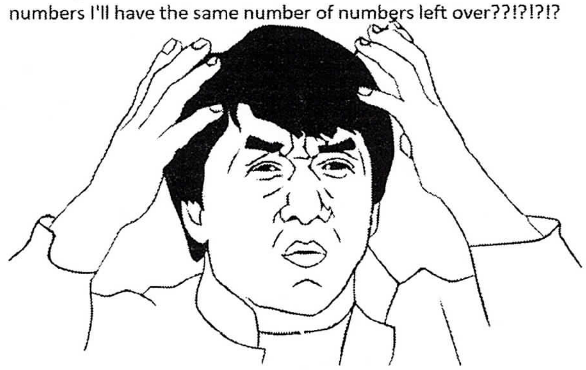
Name: SOLUTIONS

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Everything you write must be fully justified.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. Bonus problems will only be counted if all parts of all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (that's not really possible in this test anyway...)
5. Write neatly so that I am able to follow your sequence of steps.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. **Remember: if you mess up on a definition in a problem, you will get a zero for that problem.** Use the definitions from class. If you want to use another, you must first prove it is equivalent to the class' definition.
11. Other than that, have fun and good luck!

So, there are infinitely many real numbers and infinitely many integers, yet, somehow, there are more real numbers than there are integers. And if I throw away all the integers from the real numbers I'll have the same number of numbers left over??!?!?!?



SET THEORY:
Making people go crazy. Since 1891.
FML!

1. (a) (5 points) Prove $\mathbb{R}$ is uncountable. Fully justify.

Pf: It suffices to show that $\mathbb{R}$ contains an uncountable set, for then, by our theorem on denumerable sets, this would imply $\mathbb{R}$ is uncountable. We claim this is the case, in fact $(0,1) \subseteq \mathbb{R}$ is uncountable. Assume to the contrary. Then we may list all distinct elements in $(0,1)$ as:

$$a_1 = 0.a_{11}a_{12}a_{13}\ldots$$
$$a_2 = 0.a_{21}a_{22}a_{23}\ldots$$
$$a_3 = 0.a_{31}a_{32}a_{33}\ldots$$
$$\vdots$$

where the $a_{ij} \in \{0,1,\ldots,9\}$ and this list is assumed complete.

Consider the number $b = 0.b_1b_2b_3b_4\ldots \in (0,1)$ such that

$$b_i = \begin{cases} 1, & a_{ii} \neq 1 \\ 2, & a_{ii} = 1. \end{cases}$$

Then b will differ from all a_k in our list, because it will be different in the i^{th} decimal place. Hence, we've found a number not in our list, and a contradiction is obtained.

$\Rightarrow (0,1) \subseteq \mathbb{R}$ is uncountable $\Rightarrow \mathbb{R}$ is uncountable. ∎

(b) (5 points) Prove that $|\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$

Pf: Note that $|\mathbb{N}^{\mathbb{N}}| \geq |2^{\mathbb{N}}| = |\mathbb{R}|$, since $\{1,2\} \subseteq \mathbb{N}$.

Further note that the elements in $\mathbb{N}^{\mathbb{N}}$ are functions having the form $f_i = \{(1,a_{i1}),(2,a_{i2}),(3,a_{i3}),\ldots\}$ where the $a_{ij} \in \mathbb{N}$.

Define $\varphi : \mathbb{N}^{\mathbb{N}} \to (0,1)$ by $\varphi(f_i) = 0.a_{i1}a_{i2}a_{i3}a_{i4}\ldots$

φ is injective since if $\varphi(f_i) = \varphi(f_j) \Rightarrow$

$$0.a_{i1}a_{i2}a_{i3}\ldots = 0.a_{j1}a_{j2}a_{j3}\ldots$$
$$\Rightarrow a_{ik} = a_{jk} \ \forall k \in \mathbb{N} \Rightarrow f_i = f_j.$$

$\Rightarrow |\mathbb{N}^{\mathbb{N}}| \leq |(0,1)| = |\mathbb{R}|.$

Then we have $|\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$ by Shröder-Bernstein. ∎

2. (20 points) Suppose A is a non-empty set of real numbers that has a lower bound. Prove that A has an infimum, that is, a greatest lower bound. Hint: You may use the completeness axiom for $\mathbb{R}$.

Pf: Let $S \subseteq \mathbb{R}$ be a non empty set and assume $\exists m \in \mathbb{R}$ such that $m \leq s$ $\forall s \in S$, that is, S is bounded below. Consider the set $-S = \{-s \mid s \in S\}$. Note that we have $-m \geq -s$ $\forall -s \in -S$, so that $-S$ is bounded above. The completeness axiom then gives that there is a least upper bound. That is, $\exists -M \in \mathbb{R}$ s.th $-M \geq -s$ and for any other upperbound of S, $-b \in \mathbb{R}$, $-M \leq -b$.

But then we have that $M \leq s$, $\forall s \in S$ and $M \geq b$ for any b which is a lower bound of S.

That is, M is the greatest lower bound. $\blacksquare$

3. (a) (15 points) Let A and B be non-empty sets such that $|A| = |B|$. Prove that, if C is a non-mepty set, then $|C^A| = |C^B|$.

Pf: Assume $|A|=|B|$, then we have bijective functions $f: A \to B$ and $g: B \to A$. In particular, these functions are injective, and so we have $|A| \le |B|$ — ① and $|B| \le |A|$ — ②.

By part (b), inequality ① implies $|C^A| \le |C^B|$ — ③ and, inequality ② implies $|C^B| \le |C^A|$. — ④.

Applying Schröder-Bernstein to the sets C^A and C^B, inequalities ③ and ④ together imply $|C^A| = |C^B|$. ■

(b) (5 points) Assume now that $|A| \le |B|$, prove that $|C^A| \le |C^B|$.

Pf: Assume $|A| \le |B|$. Then $\exists$ an injection $\varphi: A \to B$. Set $B' = \text{range}(\varphi)$.
We will now find an injection $F: C^A \to C^B$. Index the functions in C^A by a set I and choose, and fix, any element $p \in C$. For each $f_i \in C^A$, $i \in I$, define $F(f_i) = \bar{f_i} \in C^B$ in the following way:

$$\bar{f_i}(n) = \begin{cases} f_i(\varphi^{-1}(n)) & \text{if } n \in B' \\ p & \text{if } n \notin B' \end{cases}$$

Since φ is an injective function onto B', it is bijective from $A \to B'$, and so $\varphi^{-1}(n)$ is the unique $a \in A$ such that $f(a) = n$.
This means, for $n, m \in B'$, $n \ne m$, we have $f_i(\varphi^{-1}(n)) \ne f_i(\varphi^{-1}(m))$ and so $F(f_i)(n) \ne F(f_i)(m)$. Thus F is 1-1, as desired. ■

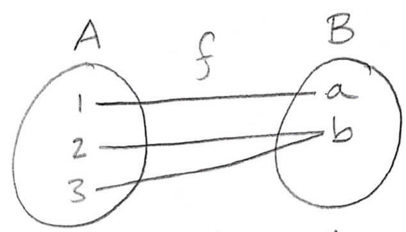
(It can be hard to visualize the construction above, so I will post another file giving concrete examples for A, B, C, so you can see how we construct these functions).

4. (20 points) Prove that if $f: A \to B$ is surjective, then there exists $g: B \to A$ such that g is injective and $g \circ f = id_A$.

Missing some conditions here. As stated, it is <u>not</u> true!

Give any counter-example!

Here's one, illustrated by a mapping diagram:



There is no injective function $g: B \to A$ s.th $g \circ f = id_A$.

5. (20 points) Let A be a set of real numbers. We say that A is *dense* (in $\mathbb{R}$) if for any two $x, y \in \mathbb{R}$ with $x < y$ there exists an $a \in A$ such that $x < a < y$. Assume A is countable. Prove that $\mathbb{R}\backslash A$ is dense.

Hint: You may use contradiction here. Assuming the contrary implies that the interval $(x, y) \subseteq A$. Explain why this must be the case under this assumption and show how this leads to a contradiction.

Pf: Assume, to the contrary, the $\mathbb{R}\backslash A$ is not dense in $\mathbb{R}$. Then $\exists x, y \in \mathbb{R}$ with $x<y$ such that there is no $a \in \mathbb{R}\backslash A$ with $x<a<y$. This means, whenever $x<b<y$, we must have $b \in A$, since it is not in $\mathbb{R}\backslash A$, by assumption. This means $(x,y) \subseteq A$, since all elements in this interval belong to A.

But, we've proven that intervals are uncountable sets if they are subsets of $\mathbb{R}$. Thus, A contains an uncountable set and so A must be uncountable. $\lightning$ $\blacksquare$

6. (a) (5 points) Prove that $\mathbb{Q}$ is countable. From scratch!

Pf: We first show that $\mathbb{Q}^+$ is countable. Note that all members of $\mathbb{Q}^+$ appear in the following table, some with repeats. Define a function f that selects the elements as shown, while skipping repeats.

Denominator \ Numerator	1	2	3	4	5 ...
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$ ...
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$ ...
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$
⋮	⋮	⋮	⋮	⋮	⋮

This defines a function

$$f: \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ q_1=1 & q_2=2 & q_3=\frac{1}{2} & q_4=3 & q_5=\frac{1}{3} & q_6=4 \cdots \end{array}$$

So that $\mathbb{Q}^+ = \{q_1, q_2, q_3, \dots\}$ and is countable, since that above function is clearly bijective.

Then we may define $g: \mathbb{N} \to \mathbb{Q}$ as follows.

$$g: \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \cdots \\ 0 & q_1 & -q_1 & q_2 & -q_2 & q_3 & -q_3 \end{array}$$

This function is also clearly bijective and so $|\mathbb{N}| = |\mathbb{Q}|$ and so $\mathbb{Q}$ is countable. ∎

(b) (5 points) Prove that for any set A, $|A| < |\mathcal{P}(A)|$.

Pf: If $A = \emptyset$, then $P(A) = \{\emptyset\} \Rightarrow |A| = 0 < 1 = |P(A)|$.
So the claim holds if $A = \emptyset$.

Now suppose $A \neq \emptyset$.

The function $f: A \to P(A)$ given by $f(a) = \{a\}$, $\forall a \in A$ is clearly an injection, since $\{a\} = \{b\} \Rightarrow a = b$.
Thus $|A| \leq |P(A)|$. It remains to show that there is no possible surjection from $A \to P(A)$.

Assume, to the contrary, that $g: A \to P(A)$ is a surjection. Set $B = \{a \in A \mid a \notin g(a)\}$, that is, the set of all elements that map to subsets not containing themselves. Clearly, $B \subseteq A$, and so there must be some $b \in A$ such that $g(b) = B$, since g is onto $P(A)$.

We have two cases: $b \in B$ or $b \notin B$.

If $b \in B$, then $b \notin g(b)$, by defn of B. But $g(b) = B \Rightarrow b \notin B = g(b)$. $\lightning$

If $b \notin B$, then $b \in g(b)$, by defn of B. But since $g(b) = B$, this means $b \in B$. $\lightning$

Therefore, there is no surjection, and so $|A| < |P(A)|$.

$\blacksquare$

Bonus Problems: (You must attempt all other parts of all other problems to be eligible. If with the bonus points, your grade exceeds 100, I'll add the excess to your first test.)

1. (10 points) (a) Let $s: \mathbb{N} \to \mathbb{R}$ be a non-decreasing sequence, that is, $s_1 \leq s_2 \leq s_3 \leq \cdots$ Suppose that the sequence is bounded above. Prove that the sequence s_n is convergent.
   Hint: It suffices to show that if $\{s_1, s_2, s_3, \ldots\}$ is an ordered, infinite set of real numbers and is bounded above by L, then $\lim_{n\to\infty} s_n = L$. This is a proofs class, so you must show a limit exists using a rigorous definition. See glossary on last page.

Pf: Let $\varepsilon > 0$. We show there exists $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - L| < \varepsilon$ for an L we will define.
Let L be the supremum of $\{s_n\}$, this exists by the completeness axiom. We claim $s_n \to L$.
Since L is the least upperbound, $L - \varepsilon$ is not an upperbound. So there exists $N \in \mathbb{N}$ s.th $L - \varepsilon < s_N$.
But this means $L - s_N < \varepsilon \Rightarrow |s_N - L| < \varepsilon$. Moreover, this will hold for all $n > N$. ∎

It may worry you that we just put absolute values in. Let this console you. Clearly,

$$L - \varepsilon < s_N < L + \varepsilon$$

not upper bound — more that least upper bound.

$$\Rightarrow -\varepsilon < s_N - L < \varepsilon$$

$$\Rightarrow |s_N - L| < \varepsilon \text{ by defn of abs. value.}$$

(b) (5 points) The above is actually a well-known theorem. What's the name of the theorem?

Monotone convergence theorem (for non-decreasing sequences).

– You can prove the result for bounded sequences similarly.

2. (15 points) Prove that, given any real number x, there exists a sequence $s: \mathbb{N} \to \mathbb{Q}$ with $s_n \to x$. (See glossary.)

For example, choose $x = \pi$, then the sequence $s = \{(1,3), (2,3.1), (3,3.14), (4,3.141), (5,3.1415), \ldots\}$ approaches π.

Hint: Use the denseness of $\mathbb{Q}$ theorem. By this theorem, there must exist rational numbers $s_1, s_2, s_3, \ldots, s_n, \ldots$ in the intervals $(x-1, x+1), \left(x-\frac{1}{2}, x+\frac{1}{2}\right), \left(x-\frac{1}{3}, x+\frac{1}{3}\right), \ldots, \left(x-\frac{1}{n}, x+\frac{1}{n}\right), \ldots$ Show that such a sequence approaches x.

Pf: It is well-known that $\frac{1}{n} \to 0$, so $\frac{1}{n}$ is eventually less than every positive real # for $n \in \mathbb{N}$ large enough. This can be proven either using limits or by applying Archimedes' property to 1 and a for $a > 0$, $a \in \mathbb{R}$.

Now, let $\varepsilon > 0$. By the above, $\exists\, n \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon$. This holds for n and any larger natural number.

Since $\frac{1}{n} > 0$ for $n \in \mathbb{N}$, we have

$$x - \frac{1}{n} < x < x + \frac{1}{n}$$

Moreover, by the denseness of $\mathbb{Q}$, for each such n, there exists $s_n \in \mathbb{Q}$ such that

$$x - \frac{1}{n} < x < s_n < x + \frac{1}{n}$$

By the first paragraph, we get

$$x - \varepsilon < x - \frac{1}{n} < x < s_n < x + \frac{1}{n} < x + \varepsilon.$$

In short, $x - \varepsilon < s_n < x + \varepsilon$.

$$\Rightarrow \quad |s_n - x| < \varepsilon$$

and so, $s_n \to x$.

$\blacksquare$

Glossary:

Definition: A (real) *sequence* is a function s from $\mathbb{N} \to \mathbb{R}$.

We denote the outputs of a sequence $s(1), s(2), s(3), \ldots$ by $s_1, s_2, s_3, \ldots$
Sometimes we define or refer to a sequence by an ordered list $\{s_1, s_2, s_3, \ldots\}$ of its outputs.
We may also denote an arbitrary term of a sequence by s_n or the entire sequence by $\{s_n\}$.

Definition: Let $L \in \mathbb{R}$. A sequence s is said to *converge to* L or *approach* L if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that, whenever $n > N$ we have $|s_n - L| < \epsilon$.

If a sequence s converges to L we write $\lim_{n \to \infty} s_n = L$ or $\lim s_n = L$ or $s_n \to L$.

Me when Jhevon says test is a take home

Me after reading through take home