Math 308 Test 1
June 28, 2018

Name: ____

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.

2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!

3. Complete all problems in the actual test. Bonus problems are, of course, optional. Bonus problems will only be counted if all other problems are attempted.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (that's not really possible in this test anyway…)

5. Write neatly so that I am able to follow your sequence of steps.

6. Read through the exam and complete the problems that are easy (for you) first!

7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.

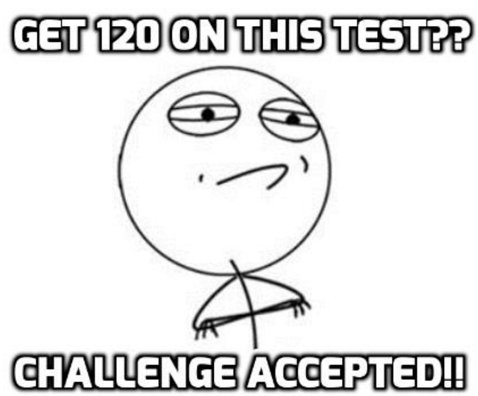
8. In fact, **cell phones should be out of sight!**

9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

10. **Remember: if you mess up on a definition in a problem, you will get a zero for that problem.** Use the definitions from class. If you want to use another, you must first prove it is equivalent to the class' definition.

11. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.



1.

(a) (10 points) Let $x, y \in \mathbb{Z}$. Prove that $(x+y)^2$ is even if and only if x and y are of the same parity.

(b) (10 points) Let $x \in \mathbb{Z}$. Prove that $3x+2$ is odd if and only if $5x+11$ is even.

2. (a) (10 points) Prove that $n! > 2^n$ for every integer $n \geq 4$.

(b) For each $n \in \mathbb{N}$, let $P(n)$: "$n^2 + 5n + 1$ is an even integer."

(i) (6 points) Prove that $P(n) \Rightarrow P(n+1)$.

(ii) (2 points) For which n is $P(n)$ actually true?

(iii) (2 points) What is the moral of this exercise?

3. (20 points) Let $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$, then $3|ab$.

Hint: You may want to prove this lemma:
Lemma: If $c \in \mathbb{Z}$, then $c^2 \equiv 0 \ (mod\ 3)$ or $c^2 \equiv 1 \ (mod\ 3)$.

Two other results from the text may come in handy (you may use these without proving them):
Result (1): If $3|x$ or $3|y$, then $3|xy$
Result (2): If 3 does not divide x, then $3|(x^2 - 1)$.

(If you choose to follow the hint, I will give you 10 points for proving the lemma and 10 points for finishing the proof of the main statement.)

4. **Definition:** Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Then we define the *supremum* of S, denoted $supS$, to be the least upper bound of S. That is, (1) $supS \geq s, \forall s \in S$, and (2) if x is another upper bound of S, then $supS \leq x$.

   (a) (10 points) Prove that if $supS \in S$, then $supS = maxS$.

   (b) **Theorem:** (Denseness of $\mathbb{Q}$) If $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

   (10 points) Prove that for any $a, b \in \mathbb{R}$, there are an infinite number of rational numbers strictly between a and b. You may or may not use the denseness of $\mathbb{Q}$ theorem stated above. You also may or may not use induction here.

5. (10 points) (a) Prove: If r be a real number such that $0 < r < 1$, then $\frac{1}{r(1-r)} \geq 4$. (You're expected to be very technical here, and will be graded accordingly.)

(b) (10 points) For sets A and B, prove that $A = (A - B) \cup (A \cap B)$.

Bonus Problems:

1. (3 points) Let x be a positive real number. Prove that $1+\frac{1}{x^4} \geq \frac{1}{x}+\frac{1}{x^3}$

2. (2 points) Let A be a set. Define a *relation on* A.

3. (5 points) Let R be a relation on a set A. Define what it means for R to be an equivalence relation on A?

4. (10 points) Define a relation R on $\mathbb{Z}$ by aRb iff $a \equiv b\ (mod\ 3)$. Show that R is an equivalence relation and find its equivalence classes.