Math 308 Test 1
June 28, 2018

Name: ______________________________SOLUTIONS____________________________________

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.

2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!

3. Complete all problems in the actual test. Bonus problems are, of course, optional. Bonus problems will only be counted if all other problems are attempted.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (that's not really possible in this test anyway…)

5. Write neatly so that I am able to follow your sequence of steps.

6. Read through the exam and complete the problems that are easy (for you) first!

7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.

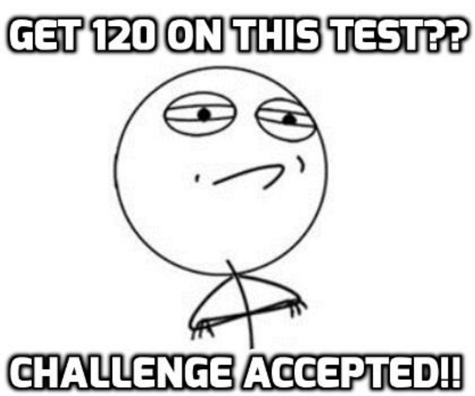
8. In fact, **cell phones should be out of sight!**

9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

10. **Remember: if you mess up on a definition in a problem, you will get a zero for that problem.** Use the definitions from class. If you want to use another, you must first prove it is equivalent to the class' definition.

11. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.



1. Solutions will vary; this is one possible way to complete this test.

(a) (10 points) Let $x, y \in \mathbb{Z}$. Prove that $(x+y)^2$ is even if and only if x and y are of the same parity.

Pf: ($\Longrightarrow$): Assume x and y are of opposite parity. WLOG, assume x is even and y is odd. Since the sum of an even integer and an odd integer is odd, we then have that $x+y$ is odd, that is, $x+y=2m+1$ for some $m \in \mathbb{Z}$. And so $(x+y)^2=(2m+1)^2=2(2m^2+2m)+1$, which is odd, and so the contrapositive holds.

($\Longleftarrow$): For the converse, assume x and y have the same parity. Then either x and y are both even or x and y are both odd. In either case, their sum is even. That is, $x+y=2k$ for some $k \in \mathbb{Z}$, and so $(x+y)^2=(2k)^2=2(2k^2)$, which is even.

■

(b) (10 points) Let $x \in \mathbb{Z}$. Prove that $3x+2$ is odd if and only if $5x+11$ is even.

Pf: ($\Longrightarrow$): Assume $3x+2$ is odd. Then we may write $3x+2=2k+1$ for some integer k. Adding $2x+9$ to both sides, we get $5x+11=2k+1+2x+9=2(k+x+5)$, which is even.

($\Longleftarrow$): Assume $5x+11$ is even. Then $5x+11=2k$ for $k \in \mathbb{Z}$. Then, adding $-2x-9$ to both sides, we get $3x+2=2k-2x-9=2(k-x-5)+1$, which is odd.

■

2. (a) (10 points) Prove that $n! > 2^n$ for every integer $n \geq 4$.

Pf: Let $P(n)$: $n! > 2^n$ for every integer $n \geq 4$. We proceed with induction on n.

For the base case: $4! = 24 > 16 = 2^4$. So $P(4)$ holds.

Assume that $P(k)$ holds for some $k \in \mathbb{N}$. We show $P(k+1)$ holds.

Since $P(k)$ holds, we have,

$$\begin{aligned} k! &> 2^k \\ \Longrightarrow (k+1)! &> (k+1) \cdot 2^k \\ &= k \cdot 2^k + 2^k \\ &\geq 4 \cdot 2^k + 2^k \\ &> 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

The above shows that $(k+1)! > 2^{k+1}$, which is $P(k+1)$, and so the claim holds by induction.

■

(b) For each $n \in \mathbb{N}$, let $P(n)$: "$n^2 + 5n + 1$ is an even integer."
(i) (6 points) Prove that $P(n) \Rightarrow P(n+1)$.

Pf: Assume that $n^2 + 5n + 1 = 2k$ for some integer k. Then we have that

$$\begin{aligned} (n+1)^2 + 5(n+1) + 1 &= n^2 + 2n + 1 + 5n + 5 + 1 \\ &= (n^2 + 5n + 1) + 2n + 6 \\ &= 2k + 2n + 6 \\ &= 2(k + n + 3) \end{aligned}$$

Which is even. Thus, $P(n+1)$ holds.

■

(ii) (2 points) For which n is $P(n)$ actually true?

None!!! The expression $n^2 + 5n + 1$ is ALWAYS odd. (To see this, if we assume n is even, that is, $n = 2k$ for an integer k, then $n^2 + 5n + 1 = 2(2k^2 + 5k) + 1$, which is odd. And if n is odd, that is, $n = 2k + 1$ for some integer k, then $n^2 + 5n + 1 = 2(2k^2 + 7k + 3) + 1$; which, again, is odd. So there is no n for which $P(n)$ is true!)

(iii) (2 points) What is the moral of this exercise?

The base case of an induction proof is important! (If one is not careful, one may have assumed from part b(i) that $n^2 + 5n + 1$ is even for all n.)

3. **(20 points) Let $\boldsymbol{a, b, c \in \mathbb{Z}}$. Prove that if $\boldsymbol{a^2 + b^2 = c^2}$, then $\boldsymbol{3|ab}$.**

Hint: You may want to prove this lemma:
Lemma: If $\boldsymbol{c \in \mathbb{Z}}$, then $\boldsymbol{c^2 \equiv 0\ (mod\ 3)}$ or $\boldsymbol{c^2 \equiv 1\ (mod\ 3)}$.

Two other results from the text may come in handy (you may use these without proving them):
Result (1): If $\boldsymbol{3|x}$ or $\boldsymbol{3|y}$, then $\boldsymbol{3|xy}$
Result (2): If 3 does not divide $\boldsymbol{x}$, then $\boldsymbol{3|(x^2 - 1)}$.

(If you choose to follow the hint, I will give you 10 points for proving the lemma and 10 points for finishing the proof of the main statement.)

$Pf\ of\ lemma$: Let $c \in \mathbb{Z}$. Then, by the division algorithm, we can have three cases:

(i) $c \equiv 0(mod\ 3)$, (ii) $c \equiv 1(mod\ 3)$, or (iii) $c \equiv 2(mod\ 3)$.

In case (i): $c = 3k$ for some integer k. And so, $c^2 = 9k^2 = 3(3k^2) \equiv 0(mod\ 3)$.
In case (ii): $c = 3k + 1$ for some integer k. Then $c^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 \equiv 1(mod\ 3)$.
In case (iii): $c = 3k + 2$ for some integer k. So $c^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 \equiv 1(mod\ 3)$.

So we see, in all cases, either $c^2 \equiv 0(mod\ 3)$ or $c^2 \equiv 1(mod\ 3)$. □

Pf: Assume, for the sake of contradiction, that $a^2 + b^2 = c^2$ but $3 \nmid ab$. Then, by the contrapositive of result (1), we have that $3 \nmid a$ and $3 \nmid b$. Result (2) then gives us that $3|(a^2 - 1)$ and $3|(b^2 - 1)$, that is, $a^2 - 1 = 3k$ and $b^2 - 1 = 3l$ for some integers k and l. In other words, $a^2 = 3k + 1$ and $b^2 = 3l + 1$. We then get that

$$c^2 = a^2 + b^2 = 3k + 1 + 3l + 1 = 3(k + l) + 2 \equiv 2(mod\ 3)$$

But this contradicts the lemma that was proven just above.

■

4. **Definition:** Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Then we define the *supremum* of S, denoted $supS$, to be the least upper bound of S. That is, (1) $supS \geq s, \forall s \in S$, and (2) if x is another upper bound of S, then $supS \leq x$.

(a) (10 points) Prove that if $supS \in S$, then $supS = maxS$.

Pf: Assume $supS \in S$. Then part (1) of the definition says that $supS \geq s, \forall s \in S$; and since $supS \in S$, we have an element in S that is greater than or equal to all other elements in S. That is, $supS$ fulfills the definition of what it means to be the maximum element of S. In other words, $supS = maxS$.

■

A nicer problem to ask would be:

Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Show that if S has a maximum element, then it must be the supremum of S. That is, $supS = maxS$.

Try to prove this on your own.

(b) **Theorem:** (Denseness of $\mathbb{Q}$) If $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

(10 points) Prove that for any $a, b \in \mathbb{R}$, there are an infinite number of rational numbers strictly between a and b. You may or may not use the denseness of $\mathbb{Q}$ theorem stated above. You also may or may not use induction here.

Pf: We will show that for any $n \in \mathbb{N}$ we can find a list of length n of distinct rationals strictly between a and b. Since $|\mathbb{N}| = \infty$, this would imply we can create an infinite list of rationals: $r_1, r_2, r_3, \dots, r_n, \dots$ between a and b. We proceed by induction on n.

Base case: clearly this holds for $n = 1$. Since, by the denseness of $\mathbb{Q}$ theorem, there exists some rational r_1 such that $a < r_1 < b$.

Assume our claim holds for $k \in \mathbb{N}$. That is, we can find $r_1, r_2, \dots, r_k \in \mathbb{Q}$ such that

$$a < r_k < r_{k-1} < \cdots < r_2 < r_1 < b$$

Then, by the denseness of $\mathbb{Q}$ property again, since $a, r_k \in \mathbb{R}$ and $a < r_k$, we can find $r_{k+1} \in \mathbb{Q}$ such that $a < r_{k+1} < r_k$. But that means we have

$$a < r_{k+1} < r_k < \cdots < r_1 < b$$

So we have a list of length $k + 1$ of distinct rationals strictly between a and b.

■

5. **(10 points) (a) Prove: If r be a real number such that $0 < r < 1$, then $\frac{1}{r(1-r)} \geq 4$. (You're expected to be very technical here, and will be graded accordingly.)**

Pf: Assume $0 < r < 1$. This means that $r > 0$ and also $r < 1 \Longrightarrow 1 - r > 0$. As the product of two positive real numbers is positive, the expression $r(1-r) > 0$. This means that we can divide by $r(1-r)$, and doing so would not affect the direction of an inequality.

Now, since $x^2 \geq 0$ for any $x \in \mathbb{R}$, we have that $(2r-1)^2 \geq 0$. This means

$$4r^2 - 4r + 1 \geq 0$$
$$\Longrightarrow 1 \geq 4r - 4r^2$$
$$\Longrightarrow 1 \geq 4r(1-r)$$
$$\Longrightarrow \frac{1}{r(1-r)} \geq 4$$

■

(b) (10 points) For sets A and B, prove that $A = (A - B) \cup (A \cap B)$.

Pf: We show that $A \subseteq (A - B) \cup (A \cap B)$ and $(A - B) \cup (A \cap B) \subseteq A$.

For the first inclusion: Assume $x \in A$. Then we have two cases, either $x \in B$ or $x \notin B$. If $x \in B$, then we have $x \in A \cap B$, which implies $x \in (A \cap B) \cup (A - B)$. If $x \notin B$, then we have $x \in A - B$, and hence $x \in (A - B) \cup (A \cap B)$. In either case, $x \in A \Longrightarrow x \in (A - B) \cup (A \cap B)$.

For the second inclusion, assume that $x \notin A$. Then it is clear that $x \notin A \cap B$ and $x \notin A - B$, since both of those sets require that $x \in A$. Since x is in none of these sets, it cannot be in any union of these sets. The reverse inclusion holds, therefore, by the contrapositive.

■

Bonus Problems:

1. **(3 points) Let x be a positive real number. Prove that $1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3}$**
   Pf: Since $x > 0$, we can divide by it. Furthermore, since $(x-1)^2 \geq 0$ in general and $x^2 + x + 1 > 0$ by our assumption, we have that $(x-1)^2(x^2 + x + 1) \geq 0$. But this means $x^4 - x^3 - x + 1 \geq 0$. In other words, $x^4 + 1 \geq x^3 + x$. Dividing both sides by x^4 gives the desired result.

   ■

2. **(2 points) Let A be a set. Define a *relation on A*.**

   Definition: A relation on A is a subset of $A \times A$ (the Cartesian product of A with itself).

3. **(5 points) Let R be a relation on a set A. Define what it means for R to be an equivalence relation on A?**

   Definition: R is called an equivalence relation iff R is reflexive, symmetric, and transitive.

   R is reflexive means: $(a, a) \in R$ for all $a \in A$.
   R is symmetric means: if $(a, b) \in R$, then $(b, a) \in R$ for $a, b \in A$.
   R is transitive means: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

4. **(10 points) Define a relation R on $\mathbb{Z}$ by aRb iff $a \equiv b \ (mod\ 3)$. Show that R is an equivalence relation and find its equivalence classes.**

   Pf: Assume $a, b, c \in \mathbb{Z}$. We show that R has the requisite properties.

   Reflexivity: Since $3|(a-a) = 0$, $a \equiv a \ (mod\ 3)$, therefore $(a, a) \in R$.

   Symmetry: Assume $a \equiv b \ (mod\ 3)$. Then $a = b + 3k$ for some integer k. This means $b = a + 3(-k)$, and so $b \equiv a \ (mod\ n)$. Thus, if $(a, b) \in R$, then $(b, a) \in R$.

   Transitivity: Assume $a \equiv b \ (mod\ 3)$ and $b \equiv c \ (mod\ 3)$. Then $a - b = 3k$ and $b - c = 3l$ for some integers k, l. Adding these two equations, we get $a - c = 3(k + l)$, so $a \equiv c \ (mod\ 3)$. This means if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

   Since R is reflexive, symmetric and transitive, it is an equivalence relation. ■

   <u>The Equivalence Classes of R:</u>

   For each $a \in \mathbb{Z}$, let $[a]$ denote its equivalence class. That is, $[a] = \{b \in \mathbb{Z} \mid aRb\} = \{b \in \mathbb{Z} \mid a \equiv b \ (mod\ 3)\}$.

$$[0] = \{0, \pm 3, \pm 6, \pm 9, \dots\} = \{3k \mid k \in \mathbb{Z}\}$$
$$[1] = \{1, -2, 4, -5, 7, -8, 10, \dots\} = \{3k + 1 \mid k \in \mathbb{Z}\}$$
$$[2] = \{2, -1, 5, -4, 8, -7, 11, \dots\} = \{3k + 2 \mid k \in \mathbb{Z}\}$$

   Since these sets partition $\mathbb{Z}$, these are all the equivalence classes of R. It has three.

   □


GOT 120?
CLOSE ENOUGH.