Math 308 - Summer 2018

Selected Solutions to HW set 8

Problems 12, 30, 38, 44, and 58 were graded for HW 8.

Disclaimer: If you have questions about any of the other problems, see me in office hours. Consider all problems important, not just the ones I provide solutions for. Also consider it important to do *more* than what is required for homework. Also note there are many ways to prove statements in general, so my proof might not look like yours, and that's fine as long as yours is correct :p

12. Let $S = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on S. Which of the properties reflexive, symmetric and transitive does the relation R posses? Justify your answers.

* R is NOT reflexive, since $(b, b) \notin R$ and $(c, c) \notin R$.
* R is NOT symmetric, since $(a, b) \in R$, but $(b, a) \notin R$ and $(a, c) \in R$ but $(c, a) \notin R$.
* R IS transitive, since $(a, b) \in R$ given that $(a, a) \& (a, b) \in R$, and $(a, b) \in R$ given that $(a, a) \& (a, b) \in R$. (In every case, including vacuously, $(x, z) \in R$ whenever $(x, y) \& (y, z) \in R$, $\forall x, y, z \in S$.

□

30. Let $H = \{2^m \mid m \in \mathbb{Z}\}$. A relation R is defined on the set $\mathbb{Q}^+$ of positive rational numbers by aRb if $a/b \in H$.

(a) Show that R is an equivalence relation.

(b) Describe the elements in the equivalence class $[3]$.

Note that, since we are dealing with $\mathbb{Q}^+$, we are never dividing by zero when computing a/b. We shall neglect to check for this in the remainder of the solution.

(a) Claim: R as defined above is an equivalence relation.

Pf: We need to show that R is reflexive, symmetric and transitive.

(i) <u>Reflexivity:</u> Since $\frac{a}{a} = 1 = 2^0$, we have $\frac{a}{a} \in H$ and so aRa.

(ii) <u>Symmetry:</u> Assume $\frac{a}{b} = 2^m$ for some $m \in \mathbb{Z}$. Then $\frac{b}{a} = 2^{-m}$, and since $-m$ is an integer whenever m is, we have that $\frac{b}{a} \in H$. Thus, $aRb \Longrightarrow bRa$.

(iii) <u>Transitivity:</u> Assume aRb and bRc, that is, $\frac{a}{b} = 2^m$ and $\frac{b}{c} = 2^n$ for integers m and n. Multiplying these two equations yields $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = 2^m \cdot 2^n = 2^{m+n}$. Since $m + n$ is an integer, $\frac{a}{c} \in H$ and so $aRb \;\&\; bRc \Longrightarrow aRc$.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation. ■

(b) By the definition of equivalence classes, we know that $[3] = \{x \in \mathbb{Q}^+ \mid xR3\}$

Here, this means $[3] = \{x \in \mathbb{Q}^+ \mid x/3 = 2^m,\ m \in \mathbb{Z}\}$

$$\Longrightarrow [3] = \{3 \cdot 2^m \mid m \in \mathbb{Z}\}$$

The above is the answer, but we will list a few elements to illustrate:

$$[3] = \{\dots, 3/8, 3/4, 3/2, 3, 6, 12, \dots\}$$

□

38. Let R be a relation defined on the set $\mathbb{N}$ by aRb if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is an equivalence relation.

Claim: R is NOT an equivalence relation.

Justification: R is reflexive and symmetric, but it is NOT transitive. Writing down all the cases for what transitivity would entail here, one would realize that some cases do not uphold transitivity. Here's an example.

Counter-example: $3R1$ and $1R5$, since 1 divides 2(3) = 6 and 1 divides 2(5) = 10, respectively. However, 3 does NOT relate to 5, since 3 does not divide 2(5) = 10 and 5 does not divide 2(3) = 6.

Thus, choosing $x = 3, y = 1, z = 5$ makes the claim xRy & $yRz \Longrightarrow xRz$ false.

There are many such examples.

□

44. Classify each of the following statements as true or false.

(a) $25 \equiv 9 \ (mod\ 8)$

(b) $-17 \equiv 9 \ (mod\ 8)$

(c) $-14 \equiv -14 \ (mod\ 4)$

(d) $25 \equiv -3 \ (mod\ 11)$.

(a) This is **TRUE**. Since $25 - 9 = 16 = 2(8)$, so $8 \mid (25 - 9) \Longrightarrow 25 \equiv 9 \ (mod\ 8)$.
(b) This is **FALSE**. Since $-17 - 9 = -26 = -4(8) + 6$, so $8 \nmid (-17 - 9) \Longrightarrow -17 \not\equiv 9 \ (mod\ 8)$.
(c) This is **TRUE**. Since $-14 - (-14) = 0 = 0(4)$, so $4 \mid \big(-14 - (-14)\big) \Longrightarrow -14 \equiv -14 \ (mod\ 4)$.
(d) This is **FALSE**. Since $25 - (-3) = 28 = 2(11) + 6$, so $11 \nmid \big(25 - (-3)\big) \Longrightarrow 25 \not\equiv -3 \ (mod\ 11)$.

□

58. Prove that multiplication in $\mathbb{Z}_n$, $n \geq 2$, defined by $[a][b] = [ab]$ is well-defined.

Pf: We need to show that the definition does not depend on the representatives chosen. That is, if $[a] = [c]$ and $[b] = [d]$, then $[a][b] = [c][d]$.

Let's show this directly. Assume $[a] = [c]$ and $[b] = [d]$. Then $a \equiv c \ (mod\ n)$ and $b \equiv d \ (mod\ n)$. In other words, we have that $a = c + nk$ and $b = d + nl$ for some integers k and l. Then we have,

$$
\begin{aligned}
[a][b] &= [ab] \\
&= [(c + nk)(d + nl)] \\
&= [cd + ncl + ndk + n^2kl] \\
&= [cd] + [n(cl + dk + nkl)] \\
&= [cd] + [0] \\
&= [cd] \\
&= [c][d]
\end{aligned}
$$

This completes the proof. ■