Math 308 - Summer 2018

Selected Solutions to HW set 6

Problems 2, 8, 20, 22, and 44 were graded for HW 6.

Disclaimer: If you have questions about any of the other problems, see me in office hours. Consider all problems important, not just the ones I provide solutions for. Also consider it important to do *more* than what is required for homework. Also note there are many ways to prove statements in general, so my proof might not look like yours, and that's fine as long as yours is correct :p

2. Prove that if A is any well-ordered set of real numbers and B is any non-empty subset of A, then B is also well-ordered.

Pf: Assume to the contrary that B is not well-ordered. Then there exists some nonempty subset $C \subset B$ that does not have a least element. But since $B \subseteq A$, we also have that $C \subset A$. But that means A contains a non-empty subset that has no least element. This contradicts the fact that A is well-ordered. ■

8. Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integers n, and then verify your formula by mathematical induction.

Pf: There are many ways to come up with a conjecture here, but let's use the "trick" we showed in class (which was also suggested in exercise 4 part (2) of this chapter).

Let $S = 1 + 4 + 7 + \cdots + (3n - 2)$. Note that $S = (3n - 2) + (3n - 5) + \cdots + 7 + 4 + 1$. Adding both these equations give

$$2S = \underbrace{(3n-1) + (3n-1) + \cdots + (3n-1)}_{n\ times} = n(3n-1)$$

$$\Rightarrow S = \frac{n(3n-1)}{2}$$

Let us conjecture this is the sum and prove it by mathematical induction.

Claim: Let $P(n): 1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$, for all $n \in \mathbb{N}$.

Pf: We employ mathematical induction.

Since $P(1): 1 = \frac{1(3(1)-1)}{2}$ is true, the base case holds.

Assume $P(k)$ holds for some $k \in \mathbb{N}$. Then this means

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}$$

$$\Rightarrow 1 + 4 + 7 + \cdots + (3k - 2) + (3(k + 1) - 2) = \frac{k(3k-1)}{2} + (3(k + 1) - 2)$$

$$= \frac{k(3k-1)}{2} + (3k + 1)$$

$$= \frac{k(3k-1)+6k+2}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Which is to say $1 + 4 + 7 + \cdots + (3k - 2) + (3(k + 1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}$.

Which is the statement $P(k + 1)$.

Since $P(k) \Longrightarrow P(k + 1)$, the claim holds by induction. ■

20. (a) Use mathematical induction to prove that every finite nonempty set of real numbers has a largest element.

(b) Use (a) to prove that every finite nonempty set of real numbers has a smallest element.

(a) Pf: Let $P(n)$: If $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and $|A| = n$ for $n \in \mathbb{N}$, then A has a largest element.

Since A has a finite number of elements, we may think of A as the set $\{a_1, a_2, \ldots, a_n\}$, where the $a_i \in \mathbb{R}$ are the members of A.

For the base case, clearly $P(1)$ holds, since if $A = \{a_1\}$, we have $a_1 \geq a \; \forall a \in A$, since $a_1 \geq a_1$.

Let us assume that any set with k elements, for some $k \in \mathbb{N}$, has a largest element. Consider the set

$A = \{a_1, a_2, \ldots, a_k\} \cup \{a_{k+1}\}$, where $a_{k+1} \neq a \; \forall a \in A$, then this is a set with $k + 1$ elements. We show that this has a largest element, assuming that every set of size k has a largest.

By our hypothesis, the set $\{a_1, a_2, \ldots, a_k\}$ has a largest element, call it b. Then we have two cases: $b \geq a_{k+1}$ or $b < a_{k+1}$. In the first case, b is the largest element of A. In the second case, a_{k+1} is the largest element of A. In either case, A has a largest element, and the claim holds by induction.

■

(b) Pf: Let $A = \{a_1, a_2, \ldots, a_n\}$ be any nonempty set of n real numbers. We will show A has a least element. Consider the set $B = \{-x \mid x \in A\}$. Since B is also a finite set of n real numbers, it has a largest element by the above result. Call the largest element $-a$; then $-a \geq -b$, for all $-b \in B$. But that means that $a \leq b$ for all $b \in A$. In other words, the set A has a least element.

■

22. Prove that $3^n > n^2$ for every positive integer n.

Pf: Let $P(n)$: $3^n > n^2$ for all $n \in \mathbb{N}$.

Base case: $P(1)$ is true, since $3^1 = 3 > 1 = 1^2$.

Inductive hypothesis: Assume that $3^k > k^2$ for some particular $k \in \mathbb{N}$, that is, $P(k)$ holds for some k.

We show $P(k + 1)$ holds.

Now, $P(k + 1)$ is the statement that $3^{k+1} > (k + 1)^2$, and note that this inequality holds if $k = 1$. So let us assume $k \geq 2$ to prove for the remaining cases.

Consider,

$$3^{k+1} = 3 \cdot 3^k$$

$$> 3 \cdot k^2 \text{ by the inductive hypothesis.}$$

$$= k^2 + k^2 + k^2$$

$$= k^2 + k \cdot k + k^2$$

$$\geq k^2 + 2k + 4 \text{ since } k \geq 2$$

$$> k^2 + 2k + 1$$

$$= (k+1)^2$$

And thus we have shown that $3^{k+1} > (k+1)^2$ for $k \geq 2$, and so $P(k+1)$ holds in all cases. Applying the conclusion of the mathematical induction principle gives the desired result.

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44. Consider the sequence $F_1, F_2, F_3, \ldots, where$

$$\boldsymbol{F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8.}$$

The terms of this sequence are called <u>Fibonacci numbers</u>.

- **(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.**
- **(b) Prove that $2|F_n$ if and only if $3|n$.**

(a) The infinite sequence of Fibonacci numbers can be defined recursively by

$F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Since, after the first two terms, each new term can be obtained by summing the previous two terms.

(b) Let $P(n)$: $2|F_n$ if and only if $3|n$, for $n \in \mathbb{N}$.

Pf: For base cases, statements $P(1)$ and $P(2)$ clearly hold. The implications, in both directions, hold true vacuously for the cases when $n = 1$ and $n = 2$.

Assume $P(i)$ holds for all values $i \geq 1$ up to some $k \in \mathbb{N}$ (we may assume, by the above $k \geq 2$, and so any integer of value $k - 1$ or larger is a member of $\mathbb{N}$). We show that $P(k+1)$ holds.

We need to show ($\Longrightarrow$): if $2|F_{k+1}$, then $3|(k+1)$, and, ($\Longleftarrow$): if $3|(k+1)$, then $2|F_{k+1}$.

($\Longrightarrow$): For the first implication, assume $3 \nmid (k+1)$, then either $k + 1 = 3m + 1$ or $k + 1 = 3m + 2$ for some integer m. In the first case, $3|k$ and so $2|F_k$ by our inductive hypothesis. And since $3|k$, we have $3 \nmid (k-1)$, and so $2 \nmid F_{k-1}$, again by our inductive hypothesis. This means F_k is even, while F_{k-1} is odd. Therefore, $F_{k+1} = F_k + F_{k-1}$ is odd, since it is the sum of an even number and an odd number. But that means $2 \nmid F_{k+1}$. In the case that $k + 1 = 3m + 2$, we have that $3 \nmid k$, but it does divide $k - 1$. By our inductive hypothesis, this means $2 \nmid F_k$ but $2|F_{k-1}$. This means F_k is odd, while F_{k-1} is even, and so $F_{k+1} = F_k + F_{k-1}$ is odd, and hence $2 \nmid F_{k+1}$. Thus, in either case, the contrapositive holds.

($\Longleftarrow$): Now let us consider the second implication. Assume $3|(k+1)$. Then $k + 1 = 3m$ for some integer m. But this means that $k = 3m - 1 = 3(m-1) + 2$ and $k - 1 = 3m - 2 = 3(m-1) + 1$, so we have that $3 \nmid k$ and $3 \nmid (k-1)$. Then, by our inductive hypothesis, $2 \nmid F_k$ and $2 \nmid F_{k-1}$. This means F_k and F_{k-1} are odd, and so $F_{k+1} = F_k + F_{k-1}$ is even. This means $2|F_{k+1}$, and so the second implication holds by direct proof.

This establishes $P(k+1)$.

■