Math 308 - Summer 2018

Selected Solutions to HW set 5

Problems 10, 16, 18, 20, and 34 were graded for HW 5.

Disclaimer: If you have questions about any of the other problems, see me in office hours. Consider all problems important, not just the ones I provide solutions for. Also consider it important to do *more* than what is required for homework. Also note there are many ways to prove statements in general, so my proof might not look like yours, and that's fine as long as yours is correct :p

10. Prove that there is no largest negative rational number.

Pf: Assume to the contrary that there is some largest negative rational number. Call this number r. Then $r < 0$ and there is no $x \in \mathbb{Q}$ such that $r < x < 0$. But consider the number $r/2$. Since r is negative and rational, so is $r/2$. But then we have $r < r/2 < 0$. So $x = r/2$ is a larger negative rational number. Contradiction.

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16. Prove that the product of an irrational number and a non-zero rational number is irrational.

Pf: Assume for the sake of contradiction that $i \in \mathbb{R} - \mathbb{Q}$ and $r \in \mathbb{Q}, r \neq 0$, but yet $ir \in \mathbb{Q}$. Then for integers a, b, c, and d with $a, b, d \neq 0$ we can write $r = \frac{a}{b}$ and $ir = \frac{c}{d}$. But this gives $i = \frac{bc}{ad}$. Since bc and ad are integers and $ad \neq 0$, we have that $i \in \mathbb{Q}$. Contradicting our assumption that it was irrational. ■

18. Let a be an irrational number and $r \neq 0$ be a rational number. Prove that if $s \in \mathbb{R}$, then either $ar + s$ or $ar - s$ is irrational.

Pf: Let us suppose that $a \in \mathbb{R} - \mathbb{Q}$ and $r \in \mathbb{Q}, r \neq 0$, and $s \in \mathbb{R}$, and furthermore both $ar + s$ and $ar - s$ are rational.

Since the sum of two rationals is rational, we have that $(ar + s) + (ar - s) = x \in \mathbb{Q}$. But that means $2ar = x$, or in other words, $ar = \frac{x}{2}$. But that means that the product of an irrational and a non-zero rational is rational. This contradicts the result proven in problem 16 above.

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20. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

Pf: We have shown in class that $\sqrt{2}$ is irrational so we shall assume this here.

Now suppose, to obtain a contradiction, $\sqrt{2} + \sqrt{3}$ is rational. Then $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. Observe that $r > 0$. But this means $\sqrt{3} = r - \sqrt{2}$. Squaring both sides we get $3 = r^2 - 2\sqrt{2}r + 2$. In other words, $\sqrt{2} = \frac{r^2-1}{2r}$. But that would mean that $\sqrt{2}$ is rational, which is absurd.

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34. Prove that if n is an odd integer, then $7n - 5$ is even, by:

(a) a direct proof

Pf: Assume n is odd. Then $n = 2k + 1$ for some integer k. But this would mean that

$$7n - 5 = 7(2k + 1) - 5 = 14k + 2 = 2(7k + 1)$$

Which is even.

■

(b) a proof by contrapositive

Pf: Assume that $7n - 5$ is odd. Then we may write $7n - 5 = 2k + 1$ for some integer k. But that means $(n + 6n) - 5 = 2k + 1$ giving that $n = 2(k - 3n + 3)$, which is even. ■

(c) a proof by contradiction

Pf: Assume, for the sake of contradiction, that n is odd, but $7n - 5$ is also odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, and we have

$$7n - 5 = 7(2k + 1) - 5 = 14k + 2 = 2(7k + 1)$$

Since this means $7n - 5$ is even, a contradiction occurs, for we assumed it was odd. ■