Math 308 - Summer 2018

Selected Solutions to HW set 4

Problems 4, 14, 32, 40, and 56 were graded for HW 4.

Disclaimer: If you have questions about any of the other problems, see me in office hours. Consider all problems important, not just the ones I provide solutions for. Also consider it important to do *more* than what is required for homework. Also note there are many ways to prove statements in general, so my proof might not look like yours, and that's fine as long as yours is correct :p

4. Let $x, y \in \mathbb{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3|(x^2 - y^2)$.

Pf: Assume that $3 \nmid x$ and $3 \nmid y$. Then we have four cases, where $k, l \in \mathbb{Z}$:

(i) $x = 3k + 1$ and $y = 3l + 1$
(ii) $x = 3k + 1$ and $y = 3l + 2$
(iii) $x = 3k + 2$ and $y = 3l + 1$
(iv) $x = 3k + 2$ and $y = 3l + 2$

Since if $3|a$, then $3|(-a)$, cases (ii) and (iii) are similar and we can show cases (i), (ii) and (iv) without loss of generality.

Case (i):
We have $x^2 - y^2 = (3k+1)^2 - (3l+1)^2 = 9k^2 + 6k + 1 - 9l^2 - 6l - 1 = 3(3k^2 - 3l^2 + 2k - 2l)$

Case (ii):
We have $x^2 - y^2 = (3k+1)^2 - (3l+2)^2 = 9k^2 + 6k + 1 - 9l^2 - 12l - 4 = 3(3k^2 - 3l^2 + 2k - 4l - 1)$

Case (iv):
We have $x^2 - y^2 = (3k+2)^2 - (3l+2)^2 = 9k^2 + 12k + 4 - 9l^2 - 12l - 4 = 3(3k^2 - 3l^2 + 4k - 4l)$

In all cases, we see that $3|(x^2 - y^2)$. ∎

14. Let $a, b, n \in \mathbb{Z}$, where $n \geq 2$. Prove that if $a \equiv b\ (mod\ n)$, then $a^2 \equiv b^2\ (mod\ n)$.

Pf: Assume that $a \equiv b\ (mod\ n)$, then $a = b + kn$ for some $k \in \mathbb{Z}$. Squaring both sides of this equation, we obtain $a^2 = b^2 + 2bkn + k^2n^2 = b^2 + (2bk + k^2n)n$. Since $2bk + k^2n$ is an integer, we get that $a^2 \equiv b^2\ (mod\ n)$. ∎

32. Recall that $\sqrt{r} > 0$ for every positive real number r.

(a) Prove that if a and b are positive real numbers, then $0 < \sqrt{ab} \leq \frac{a+b}{2}$. ($\sqrt{ab}$ and $\frac{a+b}{2}$ are called the <u>geometric mean</u> and the <u>arithmetic mean</u>, respectively, of a and b).

Pf: The first inequality is immediate, since we know $\sqrt{r} > 0$ for any positive r, then setting $r = ab$ gives the result that $0 < \sqrt{ab}$. So it remains to show the second inequality.

We may assume that $x^2 \geq 0$ for any real number x. Thus we can assume that $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$ for $a, b \in \mathbb{R}$, $a, b > 0$. But that means that $a - 2\sqrt{a}\sqrt{b} + b \geq 0$. Adding $2\sqrt{ab}$ to both sides then dividing by 2 gives the desired result.

∎

(b) Under what conditions does $\sqrt{ab} = \frac{a+b}{2}$ for positive real numbers a and b? Justify.

We only require an equality where we had an inequality above. So setting $\left(\sqrt{a} - \sqrt{b}\right)^2 = 0$ gives this result. Which means we would require that $\sqrt{a} = \sqrt{b}$ or $a = b$.

40. Let A and B be sets. Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Pf: We need to show that:

(i) $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$ and (ii) $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$.

For the first inclusion, assume $x \in A \cup B$. Then $x \in A$ or $x \in B$. Assume without loss of generality that $x \in A$. Then we have two cases, either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A \cap B$, and hence we must have that $x \in (A - B) \cup (B - A) \cup (A \cap B)$. If $x \notin B$, then $x \in A - B$, and hence $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

For the second inclusion, assume $x \notin A \cup B$. Then that means $x \notin A$ and $x \notin B$. But if x is not a member of A or B, then it cannot be a member of $A - B$, $B - A$ or $A \cap B$, and hence it is not a member of their union.

■

56. Let A, B and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Pf: We need to show inclusion in both directions.

To show that $(A - B) \cup (A - C) \subseteq A - (B \cap C)$, assume that $x \in (A - B) \cup (A - C)$. Then we have $x \in A$, but not in B, or, $x \in A$ but not in C. In both cases $x \in A$, but either $x \notin B$ or $x \notin C$. This means that $x \in A$ but $x \notin (B \cap C)$. Hence, $x \in A - (B \cap C)$.

For the reverse inclusion, assume $x \in A - (B \cap C)$. Then $x \in A$ but $x \notin (B \cap C)$. The latter means that $x \notin B$ or $x \notin C$. Assume without loss of generality that $x \notin B$. Then it means we have $x \in A$ but $x \notin B$. This means $x \in A - B$, and hence we have $x \in (A - B) \cup (A - C)$.

■