Math 308 - Summer 2018

Selected Solutions to HW set 3

Problems 10, 12, 18, 26, and 28 were graded for HW 3.

Disclaimer: If you have questions about any of the other problems, see me in office hours. Consider all problems important, not just the ones I provide solutions for. Also consider it important to do *more* than what is required for homework. Also note there are many ways to prove statements in general, so my proof might not look like yours, and that's fine as long as yours is correct :p

10. Prove that if $\boldsymbol{a}$ and $\boldsymbol{c}$ are odd integers, then $\boldsymbol{ab+bc}$ is even for every integer $\boldsymbol{b}$.

Pf: Assume that a and c are odd integers. Then $a = 2k + 1$ and $c = 2l + 1$ for some integers k and l. Then $ab + bc = b(a + c) = b(2k + 1 + 2l + 1) = 2[b(k + l + 1)]$, which is even. ■

12. Let $\boldsymbol{x} \in \mathbb{Z}$. Prove that if $\boldsymbol{2^{2x}}$ is an odd integer, then $\boldsymbol{2^{-2x}}$ is an odd integer.

Lemma 1: Let $x \in \mathbb{Z}$. If 2^{2x} is an odd integer, then $x = 0$.

$Pf\ of\ lemma\ 1$: Assume $x \neq 0$, then we have two cases: (i) $x > 0$, or (ii) $x < 0$. In the first case we have that $2^{2x} = 2 \cdot 2^{2x-1}$, and since $2x - 1$ is an integer that is at least 1, 2^{2x-1} is an integer; and so 2^{2x} is even. In the second case, 2^{2x} would not be an integer at all, in particular, it's not an odd integer. (To see this, set $x = -a$ where $a \in \mathbb{Z}^+$ and then we see that $2^{2x} = 4^x = 4^{-a} = \frac{1}{4^a} \notin \mathbb{Z}$, since 4^a would be an integer larger than 1.) In either case, 2^{2x} is not an odd integer and the contrapositive holds. □

Pf: We now prove the original claim. Assume 2^{2x} is an odd integer. Then $x = 0$ by lemma 1. But then $2^{-2x} = 2^0 = 1$, which is odd. ■

18. Let $\boldsymbol{x} \in \mathbb{Z}$. Prove that $\boldsymbol{5x - 11}$ is even if and only if $\boldsymbol{x}$ is odd.

Pf: (⇒): Assume x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$. Then $5x - 11 = 5(2k) - 11 = 2(5k - 6) + 1$, which is odd. And the forward implication holds by the contrapositive.

(⇐): Assume x is odd. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. Then $5x - 11 = 5(2k + 1) - 11 = 2(5k - 3)$, which is even. ■

26. Prove that if $\boldsymbol{n} \in \mathbb{Z}$, then $\boldsymbol{n^2 - 3n + 9}$ is odd.

Pf: We have two cases to consider here. (i) n is even, and (ii) n is odd.

If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. Then $n^2 - 3n + 9 = (2k)^2 - 3(2k) + 9 = 2(2k^2 - 3k + 4) + 1$, which is odd.

If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $n^2 - 3n + 9 = (2k + 1)^2 - 3(2k + 1) + 9 =$

$4k^2 + 4k + 1 - 6k - 3 + 9 = 2(2k^2 - k + 3) + 1$, which is odd.

Thus, in either case, $n^2 - 3n + 9$ is odd. ■

28. Let $x, y \in \mathbb{Z}$. Prove that if xy is odd, then x and y are odd.

Pf: We employ the contrapositive. Assume that x or y is even. WLOG, suppose x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$. And so $xy = (2k)y = 2(ky)$, which is even.

■