Math 201 Test 2 Review 3

December 5, 2019

Name: SOLUTIONS

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. **You have 1 hour to complete the test**.
5. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
6. Write neatly so that I am able to follow your sequence of steps and box your answers.
7. Read through the exam and complete the problems that are easy (for you) first!
8. Scientific calculators are allowed, but you are NOT allowed to use notes, graphing calculators, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
9. In fact, **cell phones should be out of sight!**
10. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
11. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.

1. Compute the following integrals (5 points each):

(a) $\int \frac{\sqrt{x} - x^2 + 2x}{x^2} dx = \int x^{-3/2} - 1 + \frac{2}{x} dx$

$= \boxed{-2x^{-1/2} - x + 2\ln|x| + C}$

(b) $\int \sin^3 x \cos^2 x \, dx = \int \sin x \sin^2 x \cos^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

$u = \cos x$

$\Rightarrow du = -\sin x \, dx$

$\Rightarrow -du = \sin x \, dx$

$\Rightarrow -\int (1-u^2) u^2 \, du$

$= \int u^4 - u^2 \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$

(c) $\int (2x-1)(x^2-1)^2 \, dx$

$= \int (2x-1)(x^4 - 2x^2 + 1) \, dx$

$= \int 2x^5 - 4x^3 + 2x - x^4 + 2x^2 - 1 \, dx$

$= \boxed{\frac{x^6}{3} - x^4 + x^2 - \frac{x^5}{5} + \frac{2}{3}x^3 - x + C}$

(d) $\displaystyle\int_1^e \frac{1}{x(1+\ln^2 x)}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

when $x=1$, $u = \ln 1 = 0$

when $x = e$, $u = \ln e = 1$

$\Rightarrow \displaystyle\int_0^1 \frac{1}{1+u^2}\,du$

$= \tan^{-1} u \Big|_0^1$

$= \tan^{-1} 1 - \tan^{-1} 0 = \boxed{\frac{\pi}{4}}$

2. (15 points) A Northman lives alongside a straight river. He wishes to build a rectangular barrier around his property to keep the white walkers out. He has 2400 feet of dragon glass fencing with which to do this, and he plans to make the river one side of the barrier (so he needs no fencing along the river). If he wants the barrier to surround the maximum area, what must the dimensions of his fence be? Use calculus to help him figure this out and defend his home. Go Team Cersei!

③ Constraint

$x + 2y = 2400$

$\Rightarrow x = 2400 - 2y$

Objective

$A = xy$

④ New objective

$A = (2400 - 2y)y$

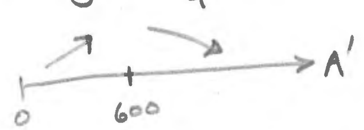
$= 2400y - 2y^2$

⑤ Max A

$A' = 0$ or ~~und~~

$\Rightarrow 2400 - 4y = 0$

$\Rightarrow y = \frac{2400}{4} = 600$



$\Rightarrow x = 2400 - 2(600)$

$= 1200$

$\Rightarrow$

⑥ Dimensions

1200 x 600 ft fence

3. (a) (10 points) A group of angry calculus students (allegedly) threw Jhevon off a cliff. Witnesses say he hit the ground after 3 seconds at a speed of 64 feet per second. Assuming the cliff is 48 feet high, acceleration due to gravity is -32 feet per second squared, and that the witnesses—instead of helping—made the effort to accurately compute the reported measurements, with what velocity was Jhevon thrown off the cliff (allegedly)? Neglect air resistance and all other factors, treat this as a freefall problem and use calculus to solve it.

$a(t) = -32$

$\Rightarrow v(t) = -32t + C$

$v(3) = -64$

$\Rightarrow -32(3) + C = -64$

$\Rightarrow C = -64 + 32(3)$

$\Rightarrow C = 32$

$\Rightarrow v(t) = -32t + 32$

$\Rightarrow v(0) = \boxed{32 \text{ ft/sec}}$

(b) (10 points) Compute $\displaystyle \frac{d}{dx}\int_{x^2}^{x^3+1} \frac{\arctan\theta}{1+\theta^3}\,d\theta$ $\displaystyle = \frac{d}{dx}\int_{c}^{x^3+1} \frac{\arctan\theta}{1+\theta^3}\,d\theta - \frac{d}{dx}\int_{c}^{x^2} \frac{\arctan\theta}{1+\theta^3}\,d\theta$

$$= \boxed{\frac{\arctan(x^3+1)}{1+(x^3+1)^3}\cdot 3x^2 - \frac{\arctan(x^2)}{1+x^6}\cdot 2x}$$

4. (5 points) Use linear approximation to approximate $\sqrt{3.99}$. Write your answer as a fraction.

Set $f(x) = \sqrt{x}$

$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Set $x = 3.99$, $a = 4$

We have,

$f(x) \approx f(a) + f'(a)(x-a)$

$\Rightarrow \sqrt{3.99} = f(3.99)$

$\approx f(4) + f'(4)(3.99-4)$

$= \sqrt{4} + \frac{1}{2\sqrt{4}}(-0.01)$

$= 2 - \frac{1}{400}$

$= \boxed{\frac{799}{400}}$

5. (a) (10 points) Find the value(s) of c guaranteed by the Mean Value Theorem for the function $f(x) = x^3 - x^2$ on the interval $[0,1]$.

What conditions allowed you to apply the Mean Value Theorem here?

We can apply the MVT since f is cont. on $[0,1]$ and differentiable on $(0,1)$. (It's a polynomial).

$f'(x) = 3x^2 - 2x$

$\Rightarrow f'(c) = 3c^2 - 2c$

Avg. R.O.C $= \frac{f(1)-f(0)}{1-0}$

$= \frac{0-0}{1}$

$= 0$

By the MVT, $\exists\, c \in (0,1)$ s.th

$f'(c) = 0$

$\Rightarrow 3c^2 - 2c = 0$

$\Rightarrow c(3c-2) = 0$

$\Rightarrow \cancel{c = 0}$ or $\boxed{c = 2/3}$

reject! Not in $(0,1)$.

(b) (10 points) Find the absolute extrema of the function $f(x)$ on the given interval.

① Crit pts

By above, $x=0$, $x = 2/3$ are crit pts.

$\Rightarrow f(0) = 0$

$f\left(\frac{2}{3}\right) = \frac{8}{27} - \frac{4}{9}$

$= -\frac{4}{27}$

② Endpoints

$f(0) = 0$

$f(1) = 0$

③ Compare

$\boxed{f\left(\frac{2}{3}\right) = -\frac{4}{27}}$ → Abs min.

$\boxed{f(0) = f(1) = 0}$ → Abs max.

6. Consider the function $f(x) = x - x^2$ on the interval $[0,3]$.

(a) (10 points) Use a finite Riemann sum with three equal subintervals and left-hand endpoints to approximate the area under $f(x)$ on the interval.

$$\Delta x = \frac{3-0}{3} = 1$$

$$\rightarrow A \approx L_3$$
$$= \left(f(0) + f(1) + f(2)\right)\Delta x$$
$$= \left(0 + 1 - 1^2 + 2 - 2^2\right)(1)$$
$$= \boxed{-2}$$

(b) (10 points) Compute the exact area under $f(x)$ on the interval by using the limit at infinity of a Riemann sum.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = \frac{3i}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n} - \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i}{n^2} - \frac{27i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{9}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

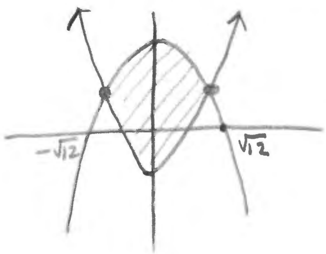
$$= \frac{9}{2} - \frac{\cancel{2}(\cancel{27})^{9}}{\cancel{6}\,\cancel{3}}$$

$$= \frac{9}{2} - 9$$

$$= \boxed{-\frac{9}{2}} \quad \text{or} \quad \boxed{-4.5}$$

Bonus Problems: 5 points each. (You must complete all problems in the actual test to be eligible). Show your work!

1. Compute the area bounded between the curves $y = 12 - x^2$ and $y = x^2 - 6$. Include a sketch of the bounded region.



Intersections

$12 - x^2 = x^2 - 6$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$.

$$A = \int \text{top} - \text{bottom}\, dx$$

$$= \int_{-3}^{3} 12 - x^2 - (x^2 - 6)\, dx$$

$$= 2\int_{0}^{3} 18 - 2x^2\, dx$$

$$= 2\left(18x - \tfrac{2}{3}x^3\right)\Big|_0^3$$

$$\rightarrow 2\left(18(3) - \tfrac{2}{3}(3)^3\right)$$

$$= \boxed{72}$$

3. Compute the average value of $f(x) = x^3 - x^2$ on the interval $[0,1]$.

$$f_{avg} = \frac{1}{1-0}\int_0^1 x^3 - x^2\, dx$$

$$= \int_0^1 x^3 - x^2\, dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{3}$$

$$= \boxed{-\frac{1}{12}}$$