Math 201 Test 1 Review 3 Solutions!!!

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. **You have 1 hour to complete the test**.
5. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
6. Write neatly so that I am able to follow your sequence of steps and box, or otherwise indicate, your answers. Solutions with no indicated answer or several contradictory answers will be considered incorrect.
7. Read through the exam and complete the problems that are easy (for you) first!
8. Scientific calculators are allowed, but you are NOT allowed to use notes, graphing calculators, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
9. In fact, **cell phones should be out of sight! If you are caught with a cellphone you will be asked to leave the exam and you'll be given a zero.**
10. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
11. Other than that, have fun and good luck!

1. (a) (15 points) Let $f(x) = 1 - \sqrt{x}$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1 - \sqrt{x+h} - (1 - \sqrt{x})}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x} + h)}{h(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x} + \sqrt{x}}$$

$$\Rightarrow \boxed{f'(x) = -\frac{1}{2\sqrt{x}}}$$

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 4$. **Write your line in $y = mx + b$ form.**

when $x = 4$, $y = 1 - \sqrt{4} = -1 \Rightarrow (x_1, y_1) = (4, -1)$

$m = f'(4) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$

$\Rightarrow$ tangent line: $y - y_1 = m(x - x_1)$

$\Rightarrow y + 1 = -\frac{1}{4}(x - 4)$

$\Rightarrow y = -\frac{1}{4}x + 1 - 1$

$\Rightarrow \boxed{y = -\frac{1}{4}x}$

2. (a) (5 points) Using an equation, define what it means for a function $f(x)$ to be continuous at a point $(a, f(a))$.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

(b) (10 points) Consider the function

$$f(x) = \begin{cases} 3, & x \le 1 \\ ax + b, & 1 < x \le 2 \\ 6 - (x-2)^2\ , & x \ge 2 \end{cases}$$

Find the values of a and b that will make the function continuous everywhere.

If $x \neq 1$ or $x \neq 2$, we're fine.

Otherwise, we need $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$ — 1

and $\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$ — 2

For 1: $a + b = 3$

For 2: $2a + b = 6$

Subtracting 1 from 2: $a = 3$

Since $a + b = 3 \Rightarrow b = 0$

$$\Rightarrow \boxed{a = 3,\ b = 0}$$

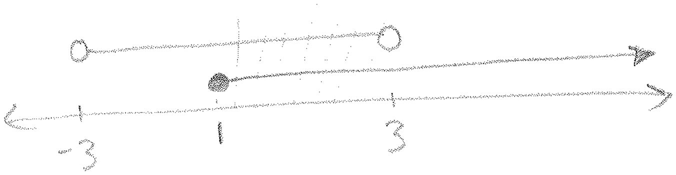
(c) (5 points) Using interval notation, state where the following function is continuous. Justify your claim!

$$g(x) = \frac{\sqrt{x-1}}{\sqrt{9-x^2}}$$

$g(x)$ is continuous as long as "$\sqrt{x-1}$" and "$\sqrt{9-x^2}$" are continuous and $\sqrt{9-x^2} \neq 0$.

$\Rightarrow x - 1 \geq 0 \Rightarrow x \geq 1$

and $9 - x^2 > 0 \Rightarrow -3 < x < 3$



$\Rightarrow$ $\boxed{g \text{ is continuous on } [1, 3)}$

3. (5 points each) Compute the following limits. **Show your work!** Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to 2}\frac{2x^2-3x-2}{x^2-x-2}$

$$= \lim_{x\to 2}\frac{(2x+1)\cancel{(x-2)}}{(x+1)\cancel{(x-2)}}$$

$$= \frac{2(2)+1}{2+1}$$

$$= \boxed{\frac{5}{3}}$$

(ii) $\lim_{x\to 3}\frac{x^2+2x-3}{x^2-4x+3}$

$$= \lim_{x\to 3}\frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x-3)}$$

$$= \lim_{x\to 3}\frac{x+3}{x-3}$$

$$\lim_{x\to 3^-}\frac{x+3}{x-3} = \frac{6}{0^-} = -\infty \qquad \lim_{x\to 3^+}\frac{x+3}{x-3} = \frac{6}{0^+} = +\infty$$

$$\Rightarrow \boxed{DNE}$$

(iii) $\lim_{x\to 1}(\underbrace{x^2-2x+1}_{(x-1)^2})\cos\frac{1}{x-1}$

If $x \neq 1$, $-1 \le \cos\frac{1}{x-1} \le 1$

$$\Rightarrow -(x-1)^2 \le (x-1)^2\cos\frac{1}{x-1} \le (x-1)^2$$

$$\Rightarrow \lim_{x\to 1} -(x-1)^2 \le \lim_{x\to 1}(x-1)^2\cos\frac{1}{x-1} \le \lim_{x\to 1}(x-1)^2$$

$$\Rightarrow 0 \le \lim_{x\to 1}(x-1)^2\cos\frac{1}{x-1} \le 0$$

$$\Rightarrow \lim_{x\to 1}(x-1)^2\cos\frac{1}{x-1} = \boxed{0}$$

By the Squeeze Theorem.

(iv) $\lim_{x\to 0}\frac{x - x\cos x}{\sin^2 2x}$

$$= \lim_{x\to 0}\frac{x}{\sin^2 2x}\cdot(1-\cos x)$$

$$= \lim_{x\to 0}\frac{4x}{4x}\cdot\frac{x}{\sin^2 2x}\cdot(1-\cos x)$$

$$= \lim_{x\to 0}\frac{1}{4}\cdot\underbrace{\left(\frac{2x}{\sin 2x}\right)^2}_{1}\cdot\underbrace{\frac{1-\cos x}{x}}_{0}$$

$$= \boxed{0}$$

(v) $\lim_{x\to\infty}\frac{e+3x-7x^4}{3x^4-2x^3+x^2+1}$

$$= \boxed{-\frac{7}{3}}$$

Ratio of leading coefficients!

4. (10 points) Use the intermediate value theorem to show that there is at least one solution to the equation $\cos x = x$. What condition(s) allow you to use the intermediate value theorem here?

Let $f(x) = \cos x - x$. It suffices to show $f(x) = 0$ somewhere.
We show it is 0 at a point in $[0, \frac{\pi}{2}]$.
Since f is continuous on $[0, \pi/2]$, the IVT applies.
Since $f(0) = 1 > 0$ and $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$, there must be an $x \in (0, \pi/2)$ such that $f(x) = 0$.

5. (5 points each) Compute and simplify $y' = \frac{dy}{dx}$ for the following:

(a) $y = \cos x + xe^x - \frac{3}{\sqrt{x}}$

$\frac{3}{\sqrt{x}} \to 3x^{-1/2}$

$\Rightarrow y' = \boxed{-\sin x + e^x + xe^x + \frac{3}{2}x^{-3/2}}$

(b) $y = \ln\sqrt{\frac{xe^x}{(x+1)^3}}$

$= \frac{1}{2}\ln\left(\frac{x \cdot e^x}{(x+1)^3}\right)$

$= \frac{1}{2}\left[\ln x + \ln e^x - 3\ln(x+1)\right]$

$\Rightarrow y' = \boxed{\frac{1}{2}\left(\frac{1}{x} + 1 - \frac{3}{x+1}\right)}$

(c) $y = 4^{x^2+1}$

$\Rightarrow y' = \boxed{(2x)4^{x^2+1}\ln 4}$

(d) $y = \frac{x^2+1}{2x-1}$

$\Rightarrow y' = \frac{(2x-1)(2x) - (x^2+1)(2)}{(2x-1)^2}$

$= \frac{4x^2 - 2x - 2x^2 - 2}{(2x-1)^2}$

$= \boxed{\frac{2x^2 - 2x - 2}{(2x-1)^2}}$

(d) $4\pi^2 - e^{2y} + \ln(x+y) = 7x$

$\Rightarrow -2y'e^{2y} + \frac{1+y'}{x+y} = 7$

$\Rightarrow -2y'e^{2y} + \frac{1}{x+y} + \frac{y'}{x+y} = 7$

$\Rightarrow y'\left(-2e^{2y} + \frac{1}{x+y}\right) = 7 - \frac{1}{x+y}$

$\Rightarrow y' = \boxed{\frac{7 - \frac{1}{x+y}}{-2e^{2y} + \frac{1}{x+y}}}$

OR

$y' = \boxed{\frac{7(x+y) - 1}{-2(x+y)e^{2y} + 1}}$

Bonus Problems: You must attempt all problems in the actual test to be eligible.

1. (5 points) Use the ϵ-δ definition of a limit to show that $\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$. Then, if $0 < |x-1| < \delta$, we have

$$\left|\frac{2x^2-x-1}{x-1} - 3\right| = \left|\frac{2x^2-x-1}{x-1} - \frac{3x-3}{x-1}\right| = \left|\frac{2x^2-4x+2}{x-1}\right| = 2\left|\frac{(x-1)^{\cancel{2}}}{\cancel{x-1}}\right| = 2|x-1| < 2\,\frac{\varepsilon}{2} = \varepsilon$$

$\blacksquare$

2. (5 points each) Compute the derivative:

(a) $y = \tan x + (\sin x)^x$

$= \tan x + e^{x \ln \sin x}$

$\Rightarrow y' = \sec^2 x + \left(\ln \sin x + x\frac{\cos x}{\sin x}\right)(\sin x)^x$

$\Rightarrow y' = \boxed{\sec^2 x + (\sin x)^x(\ln \sin x + x \cot x)}$

(b) $y = \frac{e^x\sqrt{x+1}}{x^4 \cos^2 x}$

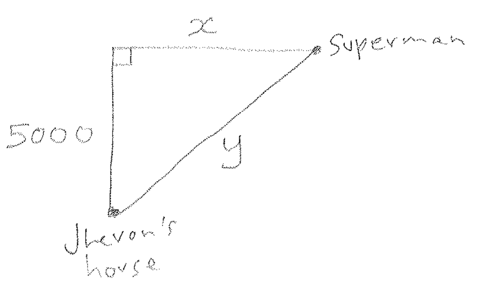
$\Rightarrow \ln y = \ln \frac{e^x\sqrt{x+1}}{x^4 \cos^2 x}$

$= x + \frac{1}{2}\ln(x+1) - 4\ln x - 2\ln\cos x$

$\Rightarrow \frac{y'}{y} = 1 + \frac{1}{2(x+1)} - \frac{4}{x} + 2\tan x$

$\Rightarrow y' = \boxed{y\left(1 + \frac{1}{2(x+1)} - \frac{4}{x} + 2\tan x\right)}$

3. (5 points) Superman is cruising at 390 feet per second at an altitude of 5000 feet and he flies directly over Jhevon's house. Assuming Superman maintains altitude and speed, how fast is the distance from the house to Superman changing at the time when Superman is 13,000 feet from the house.



Know

$\frac{dx}{dt} = 390$

Want

$\frac{dy}{dt}$ when $y = 13000$

$y^2 = x^2 + 5000^2$ — ①

$\Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

From ①, we see $x = 12000$

$\Rightarrow \frac{dy}{dt} = \frac{\cancel{12}000}{\cancel{13}000}(\cancel{390}^{30}) = \boxed{360 \text{ ft/sec}}$