Math 201 Test 1 Review 2 Solutions!!!

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. **You have 50 minutes to complete the test.**
5. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
6. Write neatly so that I am able to follow your sequence of steps and box your answers.
7. Read through the exam and complete the problems that are easy (for you) first!
8. Scientific calculators are allowed, but you are NOT allowed to use notes, graphing calculators, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
9. In fact, **cell phones should be out of sight! If you are caught with a cellphone you will be asked to leave the exam and you'll be given a zero.**
10. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
11. Other than that, have fun and good luck!

1. (a) (15 points) Let $f(x) = x - x^2$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - (x+h)^2 - (x - x^2)}{h}$$

(8)

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x^2} - 2xh - h^2 - \cancel{x} + \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{h - 2xh - h^2}{h}$$

(5)

$$= \lim_{h \to 0} \frac{\cancel{h}(1 - 2x - h)}{\cancel{h}}$$

$$\Rightarrow \boxed{f'(x) = 1 - 2x}$$ (2)

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 3$. **Write your line in $y = mx + b$ form.**

If $x_1 = 3 \Rightarrow y_1 = 3 - 3^2 = -6$

$\Rightarrow (x_1, y_1) = (3, -6)$ (2)

Also, $m = f'(3) = 1 - 2(3) = -5$ (2)

$\Rightarrow$ tangent line: $y + 6 = -5(x - 3)$

$$\Rightarrow \boxed{y = -5x + 9}$$ (1)

2. (a) (5 points) Using an equation, define what it means for a function $f(x)$ to be continuous at a point $(a, f(a))$.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$ (5) all or nothing

(b) (10 points) Consider the function

$$f(x) = \begin{cases} cx^2 + 2x & , \ x < 2 \\ x^3 - cx \ , & x \geq 2 \end{cases}$$

Find the value(s) of c that will make the function continuous everywhere.

Need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow 4c + 4 = 8 - 2c$ (5)

$\Rightarrow 6c = 4$

$\Rightarrow \boxed{c = 2/3}$ (4)

Other than at $x = 2$, we're fine. (1)

So the above makes us continuous everywhere

(c) (5 points) Using interval notation, state where the following function is continuous. Justify your claim!

$$g(x) = \frac{\sin x}{\sqrt{x^2 - 1}}$$

Since $y = \sin x$ and $y = \sqrt{x^2 - 1}$ are continuous on their domains, $g(x)$ is continuous as long as the denominator is not zero, and the radical works.

$\Rightarrow$ We need $x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0$

$\Rightarrow x < -1$ or $x > 1$

$\Rightarrow$ $\boxed{g(x) \text{ is continuous on } (-\infty, -1) \cup (1, \infty)}$

3. (5 points each) Compute the following limits. **Show your work!** Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to 0} x^4 \sin\frac{1}{x}$

If $x \neq 0$, we have

$$-1 \leq \sin\frac{1}{x} \leq 1$$

$$\Rightarrow -x^4 \leq x^4 \sin\frac{1}{x} \leq x^4$$

$$\Rightarrow \lim_{x\to 0}(-x^4) \leq \lim_{x\to 0} x^4 \sin\frac{1}{x} \leq \lim_{x\to 0} x^4$$

$$\Rightarrow 0 \leq \lim_{x\to 0} x^4 \sin\frac{1}{x} \leq 0$$

$$\Rightarrow \boxed{\lim_{x\to 0} x^4 \sin\frac{1}{x} = 0}$$ by Squeeze Theorem.

(ii) $\lim_{x\to 0} \frac{\sin x + 4x}{\tan 3x}$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{x} + \frac{4x}{x}}{\frac{\tan 3x}{x} \cdot \frac{3}{3}}$$

$$= \lim_{x\to 0} \frac{\cancel{\frac{\sin x}{x}}^{\to 1} + 4}{3 \cdot \cancel{\frac{\tan 3x}{3x}}^{\to 1}}$$

$$= \boxed{\frac{5}{3}}$$

(iii) $\lim_{x\to 1^-} \frac{x^2+3x+2}{x^2+x-2}$

$$= \lim_{x\to 1^-} \frac{(x+1)\cancel{(x+2)}}{(x-1)\cancel{(x+2)}}$$

$$= \lim_{x\to 1^-} \frac{x+1}{x-1}$$

$$= \frac{2}{0^-}$$

$$= \boxed{-\infty}$$

(iv) $\lim_{x\to -\infty} \frac{\pi + 2x - 3x^7}{2x^7 + x^3 + 3x^4}$

$$= \boxed{-\frac{3}{2}}$$

Ratio of leading coefficients.

(v) $\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3}$

$$= \lim_{x\to -1} \frac{(2x+1)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}}$$

$$= \frac{2(-1)+1}{-1-3}$$

$$= \boxed{\frac{1}{4}}$$

4. (10 points) Use the intermediate value theorem to show that there is a solution to the equation $x^4 = 3 - x$ in the interval (1,2). What condition(s) allow you to use the intermediate value theorem here?

Set $f(x) = x^4 + x - 3$. We need to show there is a $c \in (1,2)$ such that $f(c) = 0$. Since $f(x)$ is continuous (it's a polynomial) the IVT applies (3 pts). Since $f(1) = -1 < 0$ and $f(2) = 15 > 0$, there is a $c \in (1,2)$ such that $f(c) = 0$. (7 pts)

5. (5 points each) Compute and simplify $y' = \frac{dy}{dx}$ for the following:

(a) $y = x^2e^x + \frac{2}{\sqrt{x}}$ $\rightarrow 2x^{-1/2}$

$$\boxed{y' = 2xe^x + x^2e^x - x^{-3/2}}$$

(b) $y = \log_4(x^2 + 1)$

Using $\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$

$$\Rightarrow \boxed{y' = \frac{2x}{(x^2+1)\ln 4}}$$

(c) $y = \frac{(x+1)^3}{(2x+1)^5}$

Using the quotient rule:

$$y' = \frac{(2x+1)^5 \cdot 3(x+1)^2 - (x+1)^3 \cdot 5(2x+1)^4(2)}{(2x+1)^{10}}$$

$$= \frac{(x+1)^2\cancel{(2x+1)^4}[3(2x+1) - 10(x+1)]}{(2x+1)^{\cancel{10}\,6}}$$

$$\boxed{y' = \frac{(x+1)^2(-4x-7)}{(2x+1)^6}}$$

(d) $y = \ln\left[\frac{x^2e^x}{(x+1)^3}\right] = 2\ln x + x - 3\ln(x+1)$

$$\Rightarrow \boxed{y' = \frac{2}{x} + 1 - \frac{3}{x+1}}$$

(d) $e^{y^2} = 8x^2y - 3\pi$

$$\Rightarrow 2yy'e^{y^2} = 16xy + 8x^2y'$$

$$\Rightarrow y'(2ye^{y^2} - 8x^2) = 16xy$$

$$\Rightarrow \boxed{y' = \frac{16xy}{2ye^{y^2} - 8x^2}} \text{ or } = \frac{8xy}{ye^{y^2} - 4x^2}$$

Bonus Problems: You must attempt all problems in the actual test to be eligible.

1. (2 points) Use the definition of the derivative to find the derivative of $c(x) = \cos x$

$$c'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\cos x (\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \left[\cos x \cdot \overbrace{\frac{\cos h - 1}{h}}^{0} - \sin x \cdot \overbrace{\frac{\sin h}{h}}^{1} \right]$$

$$= \boxed{-\sin x}$$

2. (4 points each) Compute the derivative:

(a) $y = x^3 + (\cos x)^x$

$$= x^3 + e^{x \ln \cos x}$$

$$\Rightarrow \boxed{y' = 3x^2 + (\cos x)^x \left[\ln \cos x - x \tan x\right]}$$

(b) $y = \frac{2^x \sqrt{x-1}}{x^3 \sin x}$

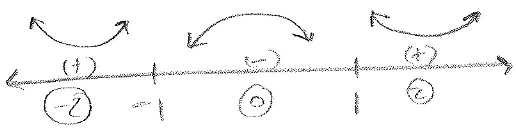
$$\Rightarrow \ln y = x \ln 2 + \tfrac{1}{2}\ln(x-1) - 3\ln x - \ln \sin x$$

$$\Rightarrow \frac{y'}{y} = \ln 2 + \frac{1}{2(x-1)} - \frac{3}{x} - \cot x$$

$$\Rightarrow \boxed{y' = \frac{2^x \sqrt{x-1}}{x^3 \sin x}\left(\ln 2 + \frac{1}{2(x-1)} - \frac{3}{x} - \cot x\right)}$$

3. (5 points) On what interval(s) is the function $f(x) = \frac{x^2}{x^2-1}$ is concave up? Concave down? Find the x-value of its inflection point, if it has one. State that it doesn't have one if that's the case. You may assume (and need not verify) that $f'(x) = -\frac{2x}{(x^2-1)^2}$ and $f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$.

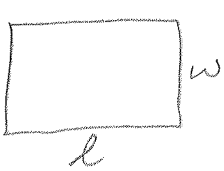
$f'' = 0$ or undefined $\Rightarrow x = \pm 1$



$\boxed{\text{C.U.: } (-\infty, -1) \cup (1, \infty) \quad \text{C.D.: } (-1, 1)}$

$\boxed{\text{No Inflection pts}}$

4. (5 points) The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?



$\frac{dl}{dt} = 8$

$\frac{dw}{dt} = 3$

$$A = lw$$

$$\Rightarrow \frac{dA}{dt} = \frac{dl}{dt} w + l \frac{dw}{dt}$$

$$= 8(10) + (20)(3)$$

$$= \boxed{140} \text{ cm}^2/\text{s}$$