Math 201 Test 1 Review 1 Solutions!!!

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. **You have 1 hour to complete the test.**
5. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
6. Write neatly so that I am able to follow your sequence of steps and box your answers.
7. Read through the exam and complete the problems that are easy (for you) first!
8. Scientific calculators are allowed, but you are NOT allowed to use notes, graphing calculators, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
9. In fact, **cell phones should be out of sight!**
10. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
11. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.

1. (a) (15 points) Let $f(x) = 2x - \frac{3}{x}$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h) - \frac{3}{x+h} - \left(2x - \frac{3}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h - \frac{3}{x+h} - \cancel{2x} + \frac{3}{x}}{h}$$

$$= \lim_{h \to 0} \left[\frac{2\cancel{h}}{\cancel{h}} + \frac{\frac{3}{x} - \frac{3}{x+h}}{h} \cdot \frac{x(x+h)}{x(x+h)} \right]$$

$$= \lim_{h \to 0} \left(2 + \frac{3(x+h) - 3x}{hx(x+h)} \right)$$

$$= \lim_{h \to 0} \left(2 + \frac{3\cancel{h}}{\cancel{h}x(x+h)} \right)$$

$$= \boxed{2 + \frac{3}{x^2}}$$

Check using power rule:

$y = 2x - 3x^{-1}$

$\Rightarrow y' = 2 + 3x^{-2}$

$= 2 + \frac{3}{x^2}$

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 1$. **Write your line in $y = mx + b$ form.**

$m = f'(1) = 2 + \frac{3}{1^2} = 5$

when $x = 1$, $y = 2(1) - \frac{3}{1} = -1$

$\Rightarrow y + 1 = 5(x - 1)$

$\Rightarrow \boxed{y = 5x - 6}$

2. (a) (5 points) Using an equation, define what it means for a function $f(x)$ to be continuous at a point $(a, f(a))$.

$$\lim_{x \to a} f(x) = f(a)$$

(b) (10 points) Consider the function

$$f(x) = \begin{cases} a - x & , \quad x \le -2 \\ \dfrac{x^2 + 4x + b}{x + 2} & , \quad x > -2 \end{cases}$$

Find the values of a and b that will make the function continuous everywhere.

We need $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = f(-2)$

$\Rightarrow a + 2 = \lim_{x \to -2^+} \dfrac{x^2 + 4x + b}{x + 2}$

$\Rightarrow$ We must have $b = 4$ * and so $\lim_{x \to -2^+} \dfrac{x^2+4x+b}{x+2} = \lim_{x \to -2^+} \dfrac{(x+2)^2}{x+2}$

$= 0$

$\Rightarrow a + 2 = 0 \Rightarrow a = -2$

$\Rightarrow \boxed{a = -2, \ b = 4}$

Other than at $x = -2$, we're OK.

* We need x^2+4x+b to factor into $(x+2)(x+c)$
$= x^2 + (2+c)x + 2c$
$\Rightarrow 2 + c = 4$
$\Rightarrow c = 2 \Rightarrow b = 2c$
$= 4.$

(c) (5 points) Using interval notation, state where the following function is continuous. Justify your claim!

$$g(x) = \frac{2x^2}{\sqrt{3 - x} - \sqrt{2 + x}}$$

Three restrictions:

① $3 - x \ge 0 \Rightarrow x \le 3$

② $2 + x \ge 0 \Rightarrow x \ge -2$

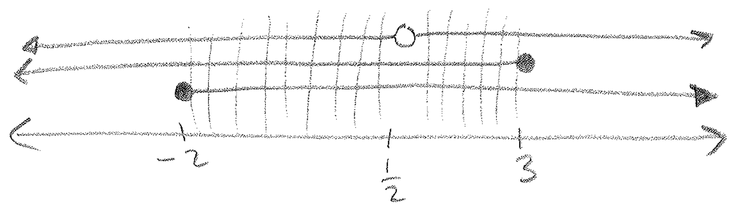
③ $\sqrt{3-x} - \sqrt{2+x} \ne 0$

$\Rightarrow \sqrt{3-x} \ne \sqrt{2+x}$

$\Rightarrow 3 - x \ne 2 + x$

$\Rightarrow 2x \ne 1$

$\Rightarrow x \ne \frac{1}{2}$

$\Rightarrow$ 

$\Rightarrow \mathrm{dom}(g) = [-2, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 3]$

$\Rightarrow \boxed{g \text{ is cont. on } [-2, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 3]}$

(It is a combination of functions cont. on their domain!)

3. (a) (3 points each) Compute the following limits, **you need not show your work for part (a)**, just state the answer. Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to-\infty} \frac{\pi^2+3x^2-3\pi x^3}{4x^3+3\pi x+2} =$ $-\frac{3\pi}{4}$

(ii) $\lim_{x\to-\infty} \frac{(2x+4)^3(2-3x^2)}{(3-x^2)(2x+1)^2} =$ $-\infty$

$8x^3\cdot(-3)x^2$

$-x^2\cdot 4x^2$

(iii) $\lim_{x\to\infty} \frac{7-\frac{2}{x}+\frac{3\pi}{x^2}+\frac{2x^7}{5}}{2\pi x^{10}+6x^3-\pi x^9} =$ 0

(iv) $\lim_{x\to-\infty} \frac{x^3\cos x^2}{4-x^3} =$ DNE

(b) (6 points each) Compute the following limits. **Show your work in this part**. Note that ∞, $-\infty$, and DNE are valid answers.

(i) $\lim_{x\to1}(x^2-2x+1)\cos^2\frac{1}{(x-1)^2}$

If $x\neq1$, then $0\le\cos^2\frac{1}{(x-1)^2}\le1$,

$\Rightarrow 0\le(x^2-2x+1)\cos^2\frac{1}{(x-1)^2}\le(x^2-2x+1)=(x-1)^2$

$\Rightarrow \lim_{x\to1}(0)\le\lim_{x\to1}(x-1)^2\cos^2\frac{1}{(x-1)^2}\le\lim_{x\to1}(x-1)^2$

$\Rightarrow 0\le\lim_{x\to1}(x-1)^2\cos^2\frac{1}{(x-1)^2}\le0$

$\boxed{\lim_{x\to1}(x-1)^2\cos^2\frac{1}{(x-1)^2}=0 \text{ by the Squeeze Theorem!}}$

(ii) $\lim_{x\to-1}\frac{|x^2-4x|}{x^2-3x-4}$

$=\lim_{x\to-1}\frac{x^2-4x}{(x-4)(x+1)}$

$=\lim_{x\to-1}\frac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)}$

$=\lim_{x\to-1}\frac{x}{x+1} = \boxed{DNE}$

(iii) $\lim_{x\to0}\frac{\sin 4x+6x}{\tan 5x}$

$=\lim_{x\to0}\frac{\frac{\sin4x}{x}+\frac{6\cancel{x}}{\cancel{x}}}{\frac{\tan5x}{x}}$

$=\lim_{x\to0}\frac{4\cdot\left(\frac{\sin4x}{4x}\right)+6}{5\cdot\left(\frac{\tan5x}{5x}\right)}$

$=\frac{10}{5}=\boxed{2}$

4. (10 points) Use the intermediate value theorem to show that there is a root of the equation $\sin x = x^2 - x$ in the interval (1,2). What condition(s) allow you to use the intermediate value theorem here?

Set $f(x) = x^2 - x - \sin x$.
It suffices to show $f(x) = 0$ for some $x \in (1,2)$.
Since f is continuous, the IVT applies.

Moreover, as
$f(1) = -\sin 1 < 0$, and
$f(2) = 2 - \sin 2 > 0$
There must be some $c \in (1,2)$ so that $f(c) = 0$.
This c is the root.

5. (4 points each) Compute $y' = \frac{dy}{dx}$ for the following:

(a) $y = 2\sqrt{x} + \frac{5}{\sqrt[3]{x}} - \pi^2$

$= 2x^{1/2} + 5x^{-1/3} - \pi^2$

$\Rightarrow \boxed{y' = x^{-1/2} - \frac{5}{3}x^{-4/3}}$

(b) $y = \log_3(x+2)$

$\Rightarrow y' = \frac{1}{(x+2)\ln 3} \cdot (1)$

$\Rightarrow \boxed{y' = \frac{1}{(x+2)\ln 3}}$

(c) $y = (x - \sin x)^{\sqrt{5}}$

$\Rightarrow \boxed{y' = \sqrt{5}(x - \sin x)^{\sqrt{5}-1}(1 - \cos x)}$

(d) $y = e^{x^2}\ln x$

$\Rightarrow \boxed{y' = 2xe^{x^2}\ln x + e^{x^2} \cdot \frac{1}{x}}$

(d) $\tan x = 8\sin y - 3xy$

$\Rightarrow \sec^2 x = 8\cos y \cdot y' - 3y - 3xy'$

$\Rightarrow \sec^2 x + 3y = y'(8\cos y - 3x)$

$\Rightarrow \boxed{y' = \frac{\sec^2 x + 3y}{8\cos y - 3x}}$

Bonus Problems: 5 points each. (You must complete all problems in the actual test to be eligible).

1. Use the definition of the derivative to find the derivative of $c(x) = \cos x$

$$c'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\cos x \underbrace{\frac{(\cos h - 1)}{h}}_{0} - \sin x \cdot \underbrace{\frac{\sin h}{h}}_{1} \right]$$

$$= \boxed{-\sin x}$$

2. Compute the derivative:

(a) $y = x^2 + x^x$

$$= x^2 + e^{x \ln x}$$

$$\Rightarrow y' = 2x + \left(\ln x + x \cdot \tfrac{1}{x}\right) e^{x \ln x}$$

$$= 2x + (\ln x + 1) x^x$$

(b) $y = \frac{e^x \sqrt{x+1}}{x^2 \sin x}$

$$\Rightarrow \ln y = \ln\left(\frac{e^x \sqrt{x+1}}{x^2 \sin x}\right)$$

$$= x + \tfrac{1}{2}\ln(x+1) - 2\ln x - \ln \sin x$$

$$\Rightarrow \frac{y'}{y} = 1 + \frac{1}{2(x+1)} - \frac{2}{x} - \cot x$$

$$\Rightarrow y' = y\left(1 + \frac{1}{2(x+1)} - \frac{2}{x} - \cot x\right)$$

3. On what interval(s) is the function $f(x) = \frac{x}{x^2-1}$ is concave up? Concave down? Find the x-value of its inflection point, if it has one. State that it doesn't have one if that's the case.

$$f' = \frac{x^2 - 1 - x(2x)}{(x^2-1)^2}$$

$$= -\frac{1+x^2}{(x^2-1)^2}$$

$$\Rightarrow f'' = -\frac{2x(x^2-1)^{\cancel{2}} - (1+x^2)(2)\cancel{(x^2-1)}(2x)}{(x^2-1)^{\cancel{4}3}}$$

$$= -\frac{2x[x^2 - 1 - 2 - 2x^2]}{(x^2-1)^3}$$

$$= \frac{2x(x^2+3)}{(x^2-1)^3}$$

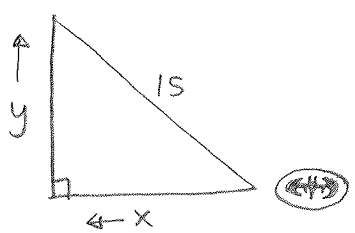
Set $f'' = 0$ or und

$x = 0$ $x = \pm 1$

f'': $(-)$ -1 $(+)$ 0 $(-)$ 1 $(+)$

$$\Rightarrow \boxed{\begin{array}{l} \text{C.U.: } (-1,0) \cup (1,\infty) \\ \text{C.D.: } (-\infty,-1) \cup (0,1) \\ \text{Inflection: } (0,0) \end{array}}$$

4. A 15 foot ladder rests against a vertical wall. Suddenly, and without warning, Batman starts pushing the foot of the ladder towards the wall at a rate of 2 meters per second. How fast is the top of the ladder sliding up the wall when the foot of the ladder is 9 feet from the wall?



$\frac{dx}{dt} = -2$

when $x = 9 \Rightarrow y = 12$.

Now $x^2 + y^2 = 15^2$

$$\Rightarrow \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = 0$$

$$\Rightarrow 9(-2) + 12 \frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{\cancel{2}(9)^3}{\cancel{12}_{\cancel{6}\,2}} = \frac{3}{2}$$

$$\boxed{\tfrac{3}{2} \text{ ft/sec}}$$